

# Dual complementary polynomials of graphs and combinatorial interpretation on the values of the Tutte polynomial at positive integers

Beifang Chen

**ABSTRACT.** We introduce a modular (integral) complementary polynomial  $\kappa(G; x, y)$  ( $\kappa_{\mathbb{Z}}(G; x, y)$ ) of two variables of a graph  $G$  by counting the number of modular (integral) complementary tension-flows (CTF) of  $G$  with an orientation  $\varepsilon$ . We study these polynomials by further introducing a cut-Eulerian equivalence relation on orientations and geometric structures such as the complementary open lattice polyhedron  $\Delta_{\text{CTF}}(G, \varepsilon)$ , the complementary open 0-1 polytope  $\Delta_{\text{CTF}}^+(G, \varepsilon)$ , and the complementary open lattice polytopes  $\Delta_{\text{CTF}}^{\rho}(G, \varepsilon)$  with respect to orientations  $\rho$ . The polynomial  $\kappa(G; x, y)$  ( $\kappa_{\mathbb{Z}}(G; x, y)$ ) is a common generalization of the modular (integral) tension polynomial  $\tau(G, x)$  ( $\tau_{\mathbb{Z}}(G, x)$ ) and the modular (integral) flow polynomial  $\varphi(G, y)$  ( $\varphi_{\mathbb{Z}}(G, y)$ ), and can be decomposed into a sum of product Ehrhart polynomials of complementary open 0-1 polytopes  $\Delta_{\text{CTF}}^+(G, \rho)$ . There are dual complementary polynomials  $\bar{\kappa}(G; x, y)$  and  $\bar{\kappa}_{\mathbb{Z}}(G; x, y)$ , dual to  $\kappa$  and  $\kappa_{\mathbb{Z}}$  respectively, in the sense that the lattice-point counting to the Ehrhart polynomials is taken inside a topological sum of the dilated closed polytopes  $\bar{\Delta}_{\text{CTF}}^+(G, \rho)$ . It turns out that the polynomial  $\bar{\kappa}(G; x, y)$  is Whitney's rank generating polynomial  $R_G(x, y)$ , which gives rise to a combinatorial interpretation on the values of the Tutte polynomial  $T_G(x, y)$  at positive integers. In particular, some special values of  $\kappa_{\mathbb{Z}}$  and  $\bar{\kappa}_{\mathbb{Z}}$  ( $\kappa$  and  $\bar{\kappa}$ ) count the number of certain special kinds (of equivalence classes) of orientations.

## 1. Introduction

The Tutte polynomial  $T_G(x, y)$  of a graph  $G$  is a common generalization of the chromatic polynomial  $\chi(G, t)$  and the flow polynomial  $\varphi(G, t)$ . Unlike the definitions of  $\chi$  by counting proper colorings and of  $\varphi$  by counting nowhere-zero flows,  $T_G$  is defined by Whitney's rank generating polynomial  $R_G(x, y)$ , rather than by counting certain combinatorial objects (see [5], p.339; [30], p.45). It has been wondered for a long time if there exists a counting style definition for  $T_G$ . In fact, the combinatorial meanings of  $T_G$  at a few special values, such as  $T_G(i, j)$  with  $1 \leq$

---

2000 *Mathematics Subject Classification.* 05A99, 05C20, 05C99, 52B40, 52C99.

*Key words and phrases.* Orientations, tension-flows, cut equivalence, Eulerian equivalence, cut-Eulerian equivalence, complementary tension-flows, complementary polyhedron, complementary polytope, complementary polynomials, dual complementary polynomials, interpretation of Tutte polynomial.

Research is supported by RGC Competitive Earmarked Research Grants 600506, 600608, and 600409.

$i, j \leq 2$ , can be easily read from  $R_G$  (see Theorem 5 in [5], p.345). However, finding combinatorial interpretations on the values of  $T_G$  at integers has been continually an active research topic since Tutte [28]. The classical interpretations of  $T_G$  were made by Tutte (see, for example, [12, 13]) as follows:

$$\tau(G, t) = (-1)^{r(G)} T_G(1 - t, 0), \quad (1.1)$$

$$\varphi(G, t) = (-1)^{n(G)} T_G(0, 1 - t), \quad (1.2)$$

where  $\tau(G, t)$  ( $= \chi(G, t)/t^{c(G)}$ ) is the tension polynomial of  $G$  and  $c(G)$  is the number of connected components. Several other combinatorial interpretations were made from various viewpoints: Crapo and Rota's finite field interpretation of  $|T_M(1 - q^k, 0)|$  on a matroid  $M$  [15]; Stanley's interpretation of  $|\chi(G, -t)| = t^{c(G)} |T_G(1 + t, 0)|$  with  $t \geq 1$  [26] and its dual version on  $|\varphi(G, -1)|$  by Green and Zaslavsky [18]; Greene's interpretation as the weight enumerator of linear codes [17] and its generalization by Barg [1] and by Green and Zaslavsky [18]; Jaeger's interpretation of linear code and dual code words [19]; Brylawski and Oxley's two-variable coloring formula [8], etc.

More recently, Kook, Reiner, and Stanton [23] found a convolution formula on the Tutte polynomial of a matroid  $M$ :

$$T_M(x, y) = \sum_{X \subseteq M} T_{M/X}(x, 0) T_{M|X}(0, y). \quad (1.3)$$

which was used by Reiner [25] to give interpretations of  $T_M$  and typically of  $T_G$  at nonpositive integers or other co-related numbers. Kochol [20, 21] introduced integral tension polynomial  $\tau_z(G, t)$  and integral flow polynomial  $\varphi_z(G, t)$ , which are closely related to  $\tau$  and  $\varphi$ , and these polynomials led him to define integral and modular tension-flow polynomials in [22] (however, his definition is problematic and one argument is incorrect; see the remark below). Gioan [16] gave combinatorial interpretations of  $T_G$  at the special integers  $(i, j)$  with  $1 \leq i, j \leq 2$ , using cycle-cocycle reversing systems. And the very recent work of Chang, Ma, and Yeh [9] on a new expression of  $T_G$ , using  $G$ -parking functions.

In the present paper, we study systematically the complementary tension-flows (CTF) of a graph  $G$  and introduce dual complementary polynomials. Fix an orientation  $\varepsilon$  (see Section 2) on  $G$  to have a digraph  $(G, \varepsilon)$  throughout. We consider functions  $h \in \mathbb{R}^E$  (which are decomposed automatically and uniquely into  $h = f + g$  of *tension-flows*  $(f, g)$ , that is,  $f$  is a tension and  $g$  is a flow of  $(G, \varepsilon)$ ) that satisfy the *complementary condition*:

$$f(e)g(e) = 0, \quad f(e) + g(e) \neq 0 \quad \text{for all } e \in E.$$

A tension-flow  $(f, g)$  is said to be a  $(p, q)$ -*tension-flow*, where  $p, q$  are positive integers, if  $|f(e)| < p$  and  $|g(e)| < q$  for all  $e \in E$ . We denote by  $K(G, \varepsilon)$  the space of all complementary tension-flows of  $(G, \varepsilon)$ , and by  $K(G, \varepsilon; p, q)$  the space of all complementary  $(p, q)$ -tension-flows.

Let  $\Delta_{\text{CTF}}(G, \varepsilon)$  denote the relatively open lattice polyhedron of all complementary  $(1, 1)$ -tension-flows of  $(G, \varepsilon)$ , and  $\Delta_{\text{CTF}}^+(G, \varepsilon)$  the relatively open 0-1 polytope of all nonnegative complementary  $(1, 1)$ -tension-flows. For each orientation  $\rho$  on  $G$ , let  $\Delta_{\text{CTF}}^\rho(G, \varepsilon)$  denote the relatively open lattice polytope of complementary  $(1, 1)$ -tension-flows  $(f, g)$  of  $(G, \varepsilon)$  such that  $f(e) + g(e) > 0$  if  $\rho(v, e) = \varepsilon(v, e)$  and  $f(e) + g(e) < 0$  if  $\rho(v, e) \neq \varepsilon(v, e)$  for each edge  $e$  at its one end-vertex  $v$ . We

shall see that  $\Delta_{\text{CTF}}(G, \varepsilon)$  is a disjoint union of  $\Delta_{\text{CTF}}^\rho(G, \varepsilon)$ , where  $\rho$  is extended over all orientations on  $G$ . Each polytope  $\Delta_{\text{CTF}}^\rho(G, \varepsilon)$  is lattice isomorphic to the 0-1 polytope  $\Delta_{\text{CTF}}^+(G, \rho)$ , and can be decomposed into a product

$$\Delta_{\text{CTF}}^+(G, \rho) = \Delta_{\text{TN}}^+(G, B_\rho) \times \Delta_{\text{FL}}^+(G, C_\rho),$$

where  $C_\rho$  is the maximal strong subdigraph of  $(G, \rho)$ ,  $B_\rho$  is the subdigraph induced by the edge set  $E - E(C_\rho)$ ,  $\Delta_{\text{TN}}^+(G, B_\rho)$  is the relatively open 0-1 polytope consisting of 1-tensions  $f$  of  $(G, \rho)$  such that  $f|_{B_\rho} > 0$  and  $f|_{C_\rho} = 0$ , and  $\Delta_{\text{FL}}^+(G, C_\rho)$  is the relatively open 0-1 polytope consisting of 1-flows  $g$  of  $(G, \rho)$  such that  $g|_{B_\rho} = 0$  and  $g|_{C_\rho} > 0$ .

Let  $\mathcal{O}(G)$  denote the set of all orientations on  $G$ . The key ingredient of the paper is to view the closure  $\bar{\Delta}_{\text{CTF}}^\rho(G, \varepsilon)$  as a dual of  $\Delta_{\text{CTF}}^\rho(G, \varepsilon)$ , and to view the topological sum

$$\tilde{\Delta}_{\text{CTF}}(G, \varepsilon) := \sum_{\rho \in \mathcal{O}(G)} \bar{\Delta}_{\text{CTF}}^\rho(G, \varepsilon) \text{ (union of disjoint copies)}$$

as a dual to the nonconvex polyhedron  $\Delta_{\text{CTF}}(G, \varepsilon)$ . We apply the Ehrhart theory to the above lattice polyhedron and polytopes.

For positive integers  $p, q$ , let  $(p, q)\Delta_{\text{CTF}}(G, \varepsilon)$  denote the dilation of  $\Delta_{\text{CTF}}(G, \varepsilon)$  in two independent parameters, consisting of the tension-flows  $(pf, qg)$  with  $(f, g) \in \Delta_{\text{CTF}}(G, \varepsilon)$ . Then  $K(G, \varepsilon; p, q) = (p, q)\Delta_{\text{CTF}}(G, \varepsilon)$ . Set

$$K_{\mathbb{Z}}(G, \varepsilon; p, q) := (\mathbb{Z}^2)^E \cap (p, q)\Delta_{\text{CTF}}(G, \varepsilon),$$

$$K_{\mathbb{Z}}^+(G, \varepsilon; p, q) := (\mathbb{Z}^2)^E \cap (p, q)\Delta_{\text{CTF}}^+(G, \varepsilon).$$

We introduce the polynomial counting functions

$$\kappa_{\mathbb{Z}}(G; p, q) := |K_{\mathbb{Z}}(G, \varepsilon; p, q)|, \quad (1.4)$$

$$\kappa_{\varepsilon}(G; p, q) := |K_{\mathbb{Z}}^+(G, \varepsilon; p, q)|. \quad (1.5)$$

For nonnegative integers  $p, q$ , let  $\tilde{K}(G, \varepsilon; p, q)$ ,  $\tilde{K}_{\mathbb{Z}}(G, \varepsilon; p, q)$  denote the topological sums of the dilations  $(p, q)\bar{\Delta}_{\text{CTF}}^\rho(G, \varepsilon)$ ,  $(\mathbb{Z}^2)^E \cap (p, q)\bar{\Delta}_{\text{CTF}}^\rho(G, \varepsilon)$ , where  $\rho \in \mathcal{O}(G)$ , respectively; set  $\bar{K}_{\mathbb{Z}}^+(G, \varepsilon; p, q) := (\mathbb{Z}^2)^E \cap (p, q)\bar{\Delta}_{\text{CTF}}^+(G, \varepsilon)$ . We introduce the dual polynomial counting functions

$$\bar{\kappa}_{\mathbb{Z}}(G; p, q) := |\tilde{K}_{\mathbb{Z}}(G, \varepsilon; p, q)|, \quad (1.6)$$

$$\bar{\kappa}_{\varepsilon}(G; p, q) := |\bar{K}_{\mathbb{Z}}^+(G, \varepsilon; p, q)|. \quad (1.7)$$

Then  $\kappa_{\mathbb{Z}}(G; p, q)$  ( $\kappa_{\varepsilon}(G; p, q)$ ) counts the number of (nonnegative) integer-valued complementary  $(p, q)$ -tension-flows of  $(G, \varepsilon)$ ;  $\bar{\kappa}_{\varepsilon}(G; p, q)$  counts the number of integer-valued tension-flows  $(f, g)$  of  $(G, \varepsilon)$  such that  $0 \leq f(e) \leq p$  and  $0 \leq g(e) \leq q$  for all  $e \in E$ ; and

$$\bar{\kappa}_{\mathbb{Z}}(G; p, q) = \sum_{\rho \in \mathcal{O}(G)} \bar{\kappa}_{\rho}(G; p, q). \quad (1.8)$$

We refer  $\kappa_{\mathbb{Z}}$  ( $\kappa_{\varepsilon}$ ) to the *integral (local) complementary polynomial* of  $G$  (with respect to  $\varepsilon$ ), and  $\bar{\kappa}_{\mathbb{Z}}$  ( $\bar{\kappa}_{\varepsilon}$ ) to the *dual integral (local) complementary polynomial*.

There is a unimodular isomorphism between  $\Delta_{\text{CTF}}^+(G, \rho)$  and  $\Delta_{\text{CTF}}^+(G, \sigma)$ , whenever  $\rho, \sigma$  differ exactly on a disjoint union of a locally directed cut and a directed Eulerian subgraph, said to be *cut-Eulerian equivalent*, denoted  $\rho \sim_{\text{CE}} \sigma$ . Indeed, the cut-Eulerian equivalence is an equivalence relation on  $\mathcal{O}(G)$ . Moreover,  $\kappa_{\rho}(G; x, y) = \kappa_{\sigma}(G; x, y)$  if  $\rho \sim_{\text{CE}} \sigma$ . Let  $[\mathcal{O}(G)]$  denote the set of cut-Eulerian

equivalence classes  $[\rho]$ , where  $\rho \in \mathcal{O}(G)$ . We define the polynomial counting functions

$$\kappa(G; p, q) := \sum_{[\rho] \in [\mathcal{O}(G)]} \kappa_\rho(G; p, q), \quad (1.9)$$

$$\bar{\kappa}(G; p, q) := \sum_{[\rho] \in [\mathcal{O}(G)]} \bar{\kappa}_\rho(G; p, q). \quad (1.10)$$

It turns out remarkably that  $\kappa(G; p, q)$  counts the number of complementary  $A$ -tensions and  $B$ -flows with abelian groups  $A, B$  of orders  $p, q$  respectively (also adopted as a definition of  $\kappa$ ), and that  $\bar{\kappa}(G; p, q)$  equals the rank generating polynomial  $R_G$ , counting the number of triples  $(\rho, f, g)$ , where  $\rho$  is a representative of a cut-Eulerian equivalence class (exactly one from each class),  $f$  is an integer-valued tension of  $(G, \rho)$  such that  $0 \leq f \leq p$ , and  $g$  is an integer-valued flow of  $(G, \rho)$  such that  $0 \leq g \leq q$ . We refer  $\kappa$  ( $\bar{\kappa}$ ) to the (*dual*) *modular complementary polynomial* of  $G$ . We summarize our main results as the following theorems.

**Theorem 1.1.** (a) *The counting function  $\kappa_{\mathbb{Z}}(G; p, q)$  ( $\bar{\kappa}_{\mathbb{Z}}(G; p, q)$ ) is a polynomial function of positive (nonnegative) integers  $p, q$ , having the same degree as the Tutte polynomial  $T_G$ , and is independent of the chosen orientation  $\varepsilon$ .*

(b) Decomposition Formulas:

$$\kappa_{\mathbb{Z}}(G; x, y) = \sum_{\rho \in \mathcal{O}(G)} \kappa_\rho(G; x, y), \quad (1.11)$$

$$\bar{\kappa}_{\mathbb{Z}}(G; x, y) = \sum_{\rho \in \mathcal{O}(G)} \bar{\kappa}_\rho(G; x, y). \quad (1.12)$$

(c) Reciprocity Laws:

$$\kappa_{\mathbb{Z}}(G; -x, -y) = \sum_{\rho \in \mathcal{O}(G)} (-1)^{r(G) + |E(C_\rho)|} \bar{\kappa}_\rho(G; x, y), \quad (1.13)$$

$$\bar{\kappa}_{\mathbb{Z}}(G; -x, -y) = \sum_{\rho \in \mathcal{O}(G)} (-1)^{r(G) + |E(C_\rho)|} \kappa_\rho(G; x, y). \quad (1.14)$$

(d) Specializations:

$$\kappa_{\mathbb{Z}}(G; x, 1) = \tau_{\mathbb{Z}}(G, x), \quad \kappa_{\mathbb{Z}}(G; 1, y) = \varphi_{\mathbb{Z}}(G, y), \quad (1.15)$$

$$\bar{\kappa}_{\mathbb{Z}}(G; x, -1) = \bar{\tau}_{\mathbb{Z}}(G, x), \quad \bar{\kappa}_{\mathbb{Z}}(G; -1, y) = \bar{\varphi}_{\mathbb{Z}}(G, y). \quad (1.16)$$

(e) Convolution Formulas:

$$\kappa_{\mathbb{Z}}(G; x, y) = \sum_{X \subseteq E} \tau_{\mathbb{Z}}(G/X, x) \varphi_{\mathbb{Z}}(G|X, y), \quad (1.17)$$

$$\bar{\kappa}_{\mathbb{Z}}(G; x, y) = \sum_{X \subseteq E} \bar{\tau}_{\mathbb{Z}}(G/X, x) \bar{\varphi}_{\mathbb{Z}}(G|X, y). \quad (1.18)$$

Equations (1.11), (1.15), and (1.17) are obtained by Kochol in a different approach by using chains and different notations (see [22], p.178).

**Theorem 1.2.** (a) *The counting function  $\kappa(G; p, q)$  ( $\bar{\kappa}(G; p, q)$ ) is a polynomial function of positive (nonnegative) integers  $p, q$ , having the same degree as the Tutte polynomial  $T_G$ , and is independent of the chosen set of distinct representatives of the cut-Eulerian equivalence classes.*

(b) Decomposition Formulas:

$$\kappa(G; x, y) = \sum_{[\rho] \in [\mathcal{O}(G)]} \kappa_\rho(G; x, y), \quad (1.19)$$

$$\bar{\kappa}(G; x, y) = \sum_{[\rho] \in [\mathcal{O}(G)]} \bar{\kappa}_\rho(G; x, y). \quad (1.20)$$

(c) Reciprocity Laws:

$$\kappa(G; -x, -y) = \sum_{[\rho] \in [\mathcal{O}(G)]} (-1)^{r(G) + |E(C_\rho)|} \bar{\kappa}_\rho(G; x, y), \quad (1.21)$$

$$\bar{\kappa}(G; -x, -y) = \sum_{[\rho] \in [\mathcal{O}(G)]} (-1)^{r(G) + |E(C_\rho)|} \kappa_\rho(G; x, y). \quad (1.22)$$

(d) Specializations:

$$\kappa(G; x, 1) = \tau(G, x), \quad \kappa(G; 1, y) = \varphi(G, y); \quad (1.23)$$

$$\bar{\kappa}(G; x, -1) = \bar{\tau}(G, x), \quad \bar{\kappa}(G; -1, y) = \bar{\varphi}(G, y). \quad (1.24)$$

(e) Convolution Formulas:

$$\kappa(G; x, y) = \sum_{X \subseteq E} \tau(G/X, x) \varphi(G|X, y), \quad (1.25)$$

$$\bar{\kappa}(G; x, y) = \sum_{X \subseteq E} \bar{\tau}(G/X, x) \bar{\varphi}(G|X, y). \quad (1.26)$$

An equivalent version of (1.19), (1.23), and (1.25) were obtained by Kochol [22] in a different approach by using chains and different notations.

**Theorem 1.3.**  $\bar{\kappa}(G; x, y) = R_G(x, y)$ .

The counting definition of  $\bar{\kappa}$  gives an immediate combinatorial interpretation on the values of the Tutte polynomial  $T_G$  at positive integers.

**Corollary 1.4.** *Let  $\text{Rep}[\mathcal{O}(G)]$  be any set of distinct representatives of cut-Eulerian equivalence classes of  $\mathcal{O}(G)$ . Then  $T_G(p, q)$ , where  $p, q$  are positive integers, counts the number of triples*

$$(\rho, f, g),$$

*where  $\rho \in \text{Rep}[\mathcal{O}(G)]$ ,  $f$  is a nonnegative integer-valued  $p$ -tension of  $(G, \rho)$ , and  $g$  is a nonnegative integer-valued  $q$ -flow of  $(G, \rho)$ .*

**REMARK.** The notion of our tension-flow is different from Kochol's tension-flow (see [22], p.177). Kochol's tension-flow on a graph  $G$  is defined as an ordered pair  $(f, g)$ , where  $f$  is a tension on  $G - H$ ,  $g$  is a flow on  $H$ , and  $H$  is a subgraph of  $G$ ; this is a wrong definition. He then identifies tensions on  $G - H$  to tensions on  $G/H$ ; this is incorrect. In fact, tensions on  $G/H$  can only be identified to tensions on  $G$  that vanish on  $H$ . However, his results can be still valid if the functions  $f$  are required to be tensions on  $G/H$  directly or tensions on  $G$  but vanishing on  $H$ . When modified in this way, then nowhere-zero tension-flows in [22] will be equivalent to our complementary tension-flows. Our definition of tension-flow is naturally formulated; names and notations are carefully selected to carry proper information for easy recognition and for keeping historic tradition. The exposition was reported in the 2007 International Conference on Graphs and Combinatorics [14].

Searching online recently we found a paper by Breuer and Sanyal [7] to interpret the values  $T_G(p+1, q+1)$  ( $=R_G(p, q)$ ) for positive integers  $p, q$  as follows:

*$R_G(p, q)$  equals the number of triples  $(f, g, \rho)$ , where  $f$  is a  $\mathbb{Z}_p$ -tension of  $(G, \varepsilon)$  and  $g$  is a  $\mathbb{Z}_q$ -flow of  $(G, \varepsilon)$  such that  $\text{supp } g \subseteq \ker f$ , and  $\rho$  is a reorientation on the edge subset  $\ker f - \text{supp } g$ .*

One might guess at first glance that this interpretation is similar or even equivalent to the result on  $\bar{\kappa}(G; p, q)$  ( $=R_G(p, q)$ ) in Corollary 1.4, but actually they are essentially different. The difference lies in the distinction between the counting over modular integers and the counting over bounded integers. The relationship between the two is interesting but highly nontrivial; it is the story on the structure of the modulo map  $M_{p,q}$ . In fact, Breuer and Sanyal's result is a reformulation of (5.3) (which is derived easily from  $R_G$ ) because, for each tension-flow  $(f, g)$  with  $\text{supp } g \subseteq \ker f$ , there are exactly  $2^{|\ker f - \text{supp } g|}$  reorientations on the edge subset  $\ker f - \text{supp } g$  as each edge has two choices to be reoriented.

## 2. Preliminaries

We follow the books [5, 6, 31] for basic concepts and notations of graphs. Let  $G = (V, E)$  be a graph with possible loops and multiple edges. Define  $r(G) := |V| - c(G)$ ,  $n(G) := |E| - r(G)$ , where  $c(G)$  is the number of connected components of  $G$ . Denote by  $\langle X \rangle$  and  $G|X$  the subgraph  $(V, X)$  induced by an edge subset  $X \subseteq E$ . An *orientation* on  $G$  is a (multivalued) function  $\varepsilon : V \times E \rightarrow \{-1, 0, 1\}$  such that (i)  $\varepsilon(v, e)$  has the ordered double-value  $\pm 1$  or  $\mp 1$  if the edge  $e$  is a loop at its end-vertex  $v$  and has a single-value otherwise, (ii)  $\varepsilon(v, e) = 0$  if  $v$  is not an end-vertex of the edge  $e$ , and (iii)  $\varepsilon(u, e)\varepsilon(v, e) = -1$  if the edge  $e$  has distinct end-vertices  $u, v$ . Pictorially, an orientation of an edge  $e$  can be expressed as an arrow from its one end-vertex  $u$  to the other end-vertex  $v$ ; such information is encoded by  $\varepsilon(u, e) = 1$  if the arrow points away from  $u$  and  $\varepsilon(v, e) = -1$  if the arrow points towards  $v$ . So every edge has exactly two orientations; each oriented edge contributes exactly one in-degree and one out-degree; and  $-(\pm 1) = \mp 1$ ,  $-(\mp 1) = \pm 1$ . A graph  $G$  with an orientation  $\varepsilon$  is referred to a *digraph*, denoted  $(G, \varepsilon)$ . We denote by  $\mathcal{O}(G)$  the set of all orientations on  $G$ .

A subgraph  $H$  of  $G$  is *Eulerian* if  $H$  has even degree at every vertex. A *circuit* is a minimal, nontrivial (=having at least one edge), Eulerian subgraph in the sense that it does not contain properly any nontrivial Eulerian subgraph. In fact, a circuit is just a closed simple path. Every nontrivial Eulerian subgraph is a disjoint union of circuits. A *directed Eulerian subgraph* is an Eulerian subgraph  $H$  together with an orientation such that at its every vertex the in-degree equals the out-degree; such an orientation is called a *local direction* of  $H$ . The orientation of a directed circuit is called a *direction* of that circuit. Any directed Eulerian subgraph is a disjoint union of directed circuits. An orientation  $\varepsilon$  on  $G$  is *acyclic* if  $(G, \varepsilon)$  has no directed circuits. We denote by  $\mathcal{O}_{\text{ac}}(G)$  the set of all acyclic orientations on  $G$ .

A *cut* of  $G$  is a nonempty edge subset  $U$  of the form  $[S, S^c]$ , where  $S$  is a nonempty proper subset of  $V$ ,  $S^c := V - S$ , and  $[S, S^c]$  is the set of all edges between vertices of  $S$  and vertices of  $S^c$ . A *bond* is a minimal cut in the sense that it does not contain any cut properly. Every cut is a disjoint union of bonds. A *directed cut* is a cut  $U = [S, S^c]$  together with an orientation on  $U$  such that the arrows of edges are all from  $S$  to  $S^c$  or all from  $S^c$  to  $S$ ; such an orientation is called a *direction* of  $U$ . A *local direction* of a cut  $U$  is an orientation  $\varepsilon_U$  on  $U$  such that

$(U, \varepsilon_U)$  is a disjoint union of directed bonds. An orientation  $\varepsilon$  on  $G$  is *totally cyclic* if  $(G, \varepsilon)$  contains no directed cuts. Totally cyclic orientations are also referred to *strong orientations*. We denote by  $\mathcal{O}_{\text{TC}}(G)$  the set of all totally cyclic orientations on  $G$ .

Given two orientations  $\rho, \sigma \in \mathcal{O}(G)$ . We say that  $\rho$  is *Eulerian-equivalent* to  $\sigma$ , denoted  $\rho \sim_{\text{EU}} \sigma$ , if the subgraph induced by the edge subset  $E(\rho \neq \sigma)$  is a directed Eulerian subgraph with the orientation  $\rho$ . Indeed,  $\sim_{\text{EU}}$  is an equivalence relation on  $\mathcal{O}(G)$ ; see [13]. If  $\rho \in \mathcal{O}_{\text{TC}}(G)$  and  $\rho \sim_{\text{EU}} \sigma$ , then  $\sigma \in \mathcal{O}_{\text{TC}}(G)$ . Analogously,  $\rho$  is said to be *cut-equivalent* to  $\sigma$ , denoted  $\rho \sim_{\text{CU}} \sigma$ , if the subgraph induced by  $E(\rho \neq \sigma)$  is a locally directed cut with the orientation  $\rho$ . Indeed,  $\sim_{\text{CU}}$  is an equivalence relation on  $\mathcal{O}(G)$ ; see [12]. If  $\rho \in \mathcal{O}_{\text{AC}}(G)$  and  $\rho \sim_{\text{CU}} \sigma$ , then  $\sigma \in \mathcal{O}_{\text{AC}}(G)$ .

Given two subgraphs  $H_i \subseteq G$  with orientations  $\varepsilon_i$ ,  $i = 1, 2$ . The *coupling* of  $\varepsilon_1$  and  $\varepsilon_2$  is a function  $[\varepsilon_1, \varepsilon_2] : E \rightarrow \{-1, 0, 1\}$ , defined for each edge  $e \in E$  (at its one end-vertex  $v$ ) by

$$[\varepsilon_1, \varepsilon_2](e) := \begin{cases} 1 & \text{if } e \in E(H_1 \cap H_2), \varepsilon_1(v, e) = \varepsilon_2(v, e), \\ -1 & \text{if } e \in E(H_1 \cap H_2), \varepsilon_1(v, e) \neq \varepsilon_2(v, e), \\ 0 & \text{otherwise.} \end{cases}$$

Now let  $(G, \varepsilon)$  be a digraph and  $A$  an abelian group throughout the whole paper. There is a *boundary operator*  $\partial_\varepsilon : A^E \rightarrow A^V$ , defined by

$$(\partial_\varepsilon f)(v) = \sum_{e \in E} \varepsilon(v, e) f(e),$$

where  $\varepsilon(v, e)$  is counted twice (as 1 and  $-1$ ) if  $e$  is a loop at its unique end-vertex  $v$ . The *flow group*  $F(G, \varepsilon; A)$  is  $\ker \partial_\varepsilon$ , whose elements are called *flows* or *A-flows* of  $(G, \varepsilon)$ . There is a *coboundary operator*  $\delta_\varepsilon : A^V \rightarrow A^E$ , defined by

$$(\delta_\varepsilon f)(e) = f(u) - f(v),$$

where  $e$  is an edge whose arrow points from one end-vertex  $u$  to the other end-vertex  $v$ . The *tension group*  $T(G, \varepsilon; A)$  is  $\text{im } \delta_\varepsilon$ , whose elements are called *tensions* or *A-tensions* of  $(G, \varepsilon)$ .

A function  $f \in A^E$  is *nowhere-zero* if  $f(e) \neq 0$  for all  $e \in E$ , and is a *q-function* if  $A = \mathbb{R}$  and  $|f(e)| < q$  for all  $e \in E$ . Let  $\tau(G, q)$  ( $\varphi(G, q)$ ) denote the number of nowhere-zero *A-tensions* (*A-flows*) of  $(G, \varepsilon)$  with  $|A| = q$ . Let  $\tau_{\mathbb{Z}}(G, q)$  ( $\varphi_{\mathbb{Z}}(G, q)$ ) denote the number of integer-valued nowhere-zero *q-tensions* (*q-flows*) of  $(G, \varepsilon)$ . Let  $\tau_\varepsilon(G, q)$  ( $\varphi_\varepsilon(G, q)$ ) denote the number of integer-valued *tensions* (*flows*)  $f$  of  $(G, \varepsilon)$  such that  $0 < f(e) < q$  for all  $e \in E$ . It is well known that  $\tau$ ,  $\varphi$ ,  $\tau_{\mathbb{Z}}$ ,  $\varphi_{\mathbb{Z}}$ ,  $\tau_\varepsilon$ , and  $\varphi_\varepsilon$  are polynomial functions of positive integers  $q$ , and that  $\tau$ ,  $\varphi$ ,  $\tau_{\mathbb{Z}}$ ,  $\varphi_{\mathbb{Z}}$  are independent of the chosen orientation  $\varepsilon$ , and that  $\tau$ ,  $\varphi$  are independent of the group structure of  $A$ . The polynomial  $\tau$  ( $\tau_{\mathbb{Z}}$ ,  $\tau_\varepsilon$ ) is referred to the *modular (integral, local) tension polynomial* of  $G$ ; and  $\varphi$  ( $\varphi_{\mathbb{Z}}$ ,  $\varphi_\varepsilon$ ) is referred to the *modular (integral, local) flow polynomial* of  $G$ ; see [12, 13] in details.

There are polynomials *dual* to  $\tau$ ,  $\varphi$ ,  $\tau_{\mathbb{Z}}$ ,  $\varphi_{\mathbb{Z}}$ ,  $\tau_\varepsilon$ , and  $\varphi_\varepsilon$  respectively. Let  $\bar{\tau}_\varepsilon(G, q)$  ( $\bar{\varphi}_\varepsilon(G, q)$ ) denote the number of integer-valued *tensions* (*flows*)  $f$  of  $(G, \varepsilon)$  such that  $0 \leq f(e) \leq q$  for all  $e \in E$ . Let  $\bar{\tau}_{\mathbb{Z}}(G, q)$  ( $\bar{\varphi}_{\mathbb{Z}}(G, q)$ ) denote the number of ordered pairs  $(\rho, f)$ , where  $\rho \in \mathcal{O}_{\text{AC}}(G)$  ( $\rho \in \mathcal{O}_{\text{TC}}(G)$ ) and  $f$  is an integer-valued *tension* (*flow*) of  $(G, \rho)$  such that  $0 \leq f(e) \leq q$  for all  $e \in E$ . Let  $\text{Rep}[\mathcal{O}_{\text{AC}}(G)]$  ( $\text{Rep}[\mathcal{O}_{\text{TC}}(G)]$ ) be any set of distinct representatives of cut-Eulerian equivalence classes of  $\mathcal{O}_{\text{AC}}(G)$  ( $\mathcal{O}_{\text{TC}}(G)$ ). Let  $\bar{\tau}(G, q)$  ( $\bar{\varphi}(G, q)$ ) denote the number of ordered

pairs  $(\rho, f)$ , where  $\rho \in \text{Rep}[\mathcal{O}_{\text{AC}}(G)]$  ( $\text{Rep}[\mathcal{O}_{\text{TC}}(G)]$ ) and  $f$  is an integer-valued tension (flow) of  $(G, \rho)$  such that  $0 \leq f(e) \leq q$  for all  $e \in E$ . It turns out that  $\bar{\tau}_\varepsilon, \bar{\varphi}_\varepsilon, \bar{\tau}_\mathbb{Z}, \bar{\varphi}_\mathbb{Z}, \bar{\tau}$ , and  $\bar{\varphi}$  are polynomial functions of nonnegative integers  $q$ , and are independent of the chosen set of distinct representatives. Moreover, these polynomials represent the same polynomials  $\tau_\varepsilon, \varphi_\varepsilon, \tau_\mathbb{Z}, \varphi_\mathbb{Z}, \tau$ , and  $\varphi$  respectively, up to sign and a change of the variable; see [12, 13] in details.

Let  $A, B$  be abelian groups. The *tension-flow group* of the digraph  $(G, \varepsilon)$  is the abelian group

$$\Omega(G, \varepsilon; A, B) := T(G, \varepsilon; A) \times F(G, \varepsilon; B),$$

whose elements are called *tension-flows*. A tension-flow  $(f, g)$  is said to be *nowhere-zero* if

$$(f(e), g(e)) \neq (0, 0) \quad \text{for all } e \in E,$$

and to be *complementary* if  $\ker f = \text{supp } g$ , where

$$\ker f := \{e \in E \mid f(e) = 0\}, \quad \text{supp } g := \{e \in E \mid g(e) \neq 0\}.$$

Let  $K(G, \varepsilon; A, B)$  denote the set of all complementary tension-flows of  $(G, \varepsilon)$ . For simplicity, we write  $F(G, \varepsilon)$  for  $F(G, \varepsilon; \mathbb{R})$ ,  $T(G, \varepsilon)$  for  $T(G, \varepsilon; \mathbb{R})$ , and  $\Omega(G, \varepsilon)$  for  $\Omega(G, \varepsilon; \mathbb{R})$ . It is well known that  $T(G, \varepsilon)$  and  $F(G, \varepsilon)$  are orthogonal complements in the Euclidean space  $\mathbb{R}^E$ . For positive integer  $p, q$ , a real-valued tension-flow  $(f, g)$  is called a  $(p, q)$ -*tension-flow* if  $|f(e)| < p$  and  $|g(e)| < q$  for all  $e \in E$ .

The present paper is conceptually rely on the Ehrhart theory of lattice polytopes and polyhedra for which we refer to [4, 2, 10, 11, 27]. Let  $X$  be a bounded lattice polyhedron (finite disjoint union of relatively open convex lattice polytopes) in the Euclidean  $d$ -space  $\mathbb{R}^d$ . Set  $qX := \{qx \mid x \in X\}$  for positive integers  $q$ . Then the counting function

$$L(P, q) := \#(\mathbb{Z}^d \cap qX) \tag{2.1}$$

is a polynomial function of positive integers  $q$  of degree  $\dim X$ , called the *Ehrhart polynomial* of  $X$ . If  $P$  is a relatively open lattice polytope and  $\bar{P}$  its closure, then  $L(P, t)$  and  $L(\bar{P}, t)$  satisfy the *Reciprocity Law*:

$$L(P, -t) = (-1)^{\dim P} L(\bar{P}, t). \tag{2.2}$$

Moreover,  $L(P, 0) = (-1)^{\dim P}$ ,  $L(\bar{P}, 0) = 1$ .

### 3. Integral complementary polynomials

Recall that a real-valued tension-flow  $(f, g) \in \Omega(G, \varepsilon)$  is *complementary* if and only if  $f(e)g(e) = 0$  and  $f(e) + g(e) \neq 0$  for all  $e \in E$ . We denote by  $K(G, \varepsilon)$  the set of all real-valued complementary tension-flows of  $(G, \varepsilon)$ . We introduce the *complementary polyhedron*

$$\Delta_{\text{CTF}}(G, \varepsilon) := \{(f, g) \in K(G, \varepsilon) : 0 < |f(e) + g(e)| < 1, e \in E\} \tag{3.1}$$

which is a bounded relatively open nonconvex polyhedron, the *complementary polytope*

$$\Delta_{\text{CTF}}^+(G, \varepsilon) := \{(f, g) \in K(G, \varepsilon) \mid 0 < f(e) + g(e) < 1, e \in E\} \tag{3.2}$$

which is a relatively open 0-1 convex polytope, and the relatively open convex polytope (with respect to an orientation  $\rho$ )

$$\Delta_{\text{CTF}}^\rho(G, \varepsilon) := \{(f, g) \in \Delta_{\text{CTF}}(G, \varepsilon) : [\rho, \varepsilon](e)(f + g)(e) > 0, e \in E\}. \tag{3.3}$$



For positive integers  $p, q$ , recall the counting function

$$\kappa_{\mathbb{Z}}(G; p, q) := \#\{\mathbb{Z}\text{-valued complementary } (p, q)\text{-tension-flows of } (G, \varepsilon)\}. \quad (3.4)$$

For nonnegative integers  $p, q$ , recall the counting function

$$\begin{aligned} \bar{\kappa}_{\mathbb{Z}}(G; p, q) := \#\{(\rho, f, g) \mid \rho \in \mathcal{O}(G), (f, g) \in (\mathbb{Z}^2)^E \cap \Omega(G, \varepsilon) \\ \text{such that } 0 \leq f(e) \leq p, 0 \leq g(e) \leq q \text{ for } e \in E\}. \end{aligned} \quad (3.5)$$

Let  $\tilde{\Delta}_{\text{CTF}}(G, \varepsilon) = \sum_{\rho \in \mathcal{O}(G)} \bar{\Delta}_{\text{CTF}}^{\rho}(G, \varepsilon)$  be a topological sum defined as a disjoint union of copies of the closures  $\bar{\Delta}_{\text{CTF}}^{\rho}(G, \varepsilon)$ , one copy for each  $\rho \in \mathcal{O}(G)$ . Then  $\bar{\kappa}_{\mathbb{Z}}(G; p, q)$  counts the number of lattice points of  $(p, q)\tilde{\Delta}_{\text{CTF}}(G, \varepsilon)$ .

We introduce the following two special directed subgraphs of  $(G, \varepsilon)$ :

$$\begin{aligned} B_{\varepsilon} &:= \text{union of directed bonds of } (G, \varepsilon), \\ C_{\varepsilon} &:= \text{union of directed circuits of } (G, \varepsilon). \end{aligned}$$

It is clear that  $B_{\varepsilon}$  is acyclic,  $C_{\varepsilon}$  is totally cyclic, and their edge sets are disjoint. The following lemma is a special case of Minty's Colored Arc Lemma [24].

**Lemma 3.1.**  $E = E(B_{\varepsilon}) \sqcup E(C_{\varepsilon})$  (disjoint union).

PROOF. Since each directed circuit of  $(G, \varepsilon)$  is edge-disjoint from any directed bond of  $(G, \varepsilon)$ , it is clear that the edge sets of  $B_{\varepsilon}$  and  $C_{\varepsilon}$  are disjoint. To see that  $E(B_{\varepsilon}) = E - E(C_{\varepsilon})$ , consider the quotient digraph  $G/C_{\varepsilon}$  obtained from  $(G, \varepsilon)$  by contracting the edges of  $C_{\varepsilon}$ . Clearly,  $G/C_{\varepsilon}$  is acyclic and the edge set of  $G/C_{\varepsilon}$  can be identified as  $E - E(C_{\varepsilon})$ . It is clear that  $E(G/C_{\varepsilon})$  can be written as a union of directed bonds (not necessarily disjoint). For ease of discussion, we call the inverse operation of contracting an edge as a *blow-up* at a vertex. It is easy to see that blow-up does not change directed bonds. So every directed bond of  $G/C_{\varepsilon}$  is preserved into a directed bond in  $(G, \varepsilon)$  when the edges of  $C_{\varepsilon}$  are blew up from  $G/C_{\varepsilon}$ . So  $E - E(C_{\varepsilon})$  is a union of directed bonds. Hence  $E(B_{\varepsilon}) = E - E(C_{\varepsilon})$ .  $\square$

Recall that the tension polytope and the flow polytope of the digraph  $(G, \varepsilon)$ , which are relatively open 0-1 polytopes (see [12, 13]), are defined as

$$\begin{aligned} \Delta_{\text{TN}}^{+}(G, \varepsilon) &:= \{f \in T(G, \varepsilon) \mid 0 < f(e) < 1, e \in E\}, \\ \Delta_{\text{FL}}^{+}(G, \varepsilon) &:= \{f \in F(G, \varepsilon) \mid 0 < f(e) < 1, e \in E\}. \end{aligned}$$

The complementary polytope  $\Delta_{\text{CTF}}^{+}(G, \varepsilon)$  can be decomposed into a product of a face of  $\Delta_{\text{TN}}^{+}(G, \varepsilon)$  and a face of  $\Delta_{\text{FL}}^{+}(G, \varepsilon)$ . In fact,

$$\Delta_{\text{CTF}}^{+}(G, \varepsilon) = \Delta_{\text{TN}}^{+}(G, B_{\varepsilon}) \times \Delta_{\text{FL}}^{+}(G, C_{\varepsilon}), \quad (3.6)$$

where

$$\begin{aligned} \Delta_{\text{TN}}^{+}(G, B_{\varepsilon}) &:= \{f \in T(G, \varepsilon) \mid 0 < f(e) < 1, e \in B_{\varepsilon}, f|_{C_{\varepsilon}} = 0\}, \\ \Delta_{\text{FL}}^{+}(G, C_{\varepsilon}) &:= \{g \in F(G, \varepsilon) \mid 0 < g(e) < 1, e \in C_{\varepsilon}, g|_{B_{\varepsilon}} = 0\}. \end{aligned}$$

To find the relationship holding among the associated polyhedra and polytopes, we need the following involution

$$P_{\rho, \sigma} : \mathbb{R}^E \rightarrow \mathbb{R}^E, \quad f \mapsto [\rho, \sigma] f$$

associated with two orientations  $\rho, \sigma \in \mathcal{O}(G)$ .

**Lemma 3.2.** (a)  $\Delta_{\text{CTF}}(G, \varepsilon) = \bigsqcup_{\rho \in \mathcal{O}(G)} \Delta_{\text{CTF}}^\rho(G, \varepsilon)$ .  
 (b)  $\Delta_{\text{CTF}}^\rho(G, \varepsilon) = P_{\rho, \varepsilon} \Delta_{\text{CTF}}^+(G, \rho)$ .  
 (c)  $(p, q) \Delta_{\text{CTF}}^+(G, \varepsilon) = p \Delta_{\text{TN}}^+(G, B_\varepsilon) \times q \Delta_{\text{FL}}^+(G, C_\varepsilon)$ .  
 (d)  $p \Delta_{\text{TN}}^+(G, B_\varepsilon) \simeq p \Delta_{\text{TN}}^+(G/C_\varepsilon, \varepsilon)$ ,  $q \Delta_{\text{FL}}^+(G, C_\varepsilon) \simeq q \Delta_{\text{FL}}^+(G|C_\varepsilon, \varepsilon)$ ; the isomorphisms send lattice points to lattice points.

PROOF. (a) The right-hand side is clearly contained in the left-hand side, since each element in the right-hand side is a real-valued complementary  $(1, 1)$ -tension-flow. Conversely, for each real-valued complementary  $(1, 1)$ -tension-flow  $(f, g)$  of  $(G, \varepsilon)$ , let  $\rho$  be the orientation defined by

$$\rho(v, e) := \begin{cases} \varepsilon(v, e) & \text{if } f(e) + g(e) > 0, \\ -\varepsilon(v, e) & \text{if } f(e) + g(e) < 0, \end{cases}$$

where  $v$  is an end-vertex of the edge  $e \in E$ . It is clear that  $[\rho, \varepsilon](f + g)(e) > 0$  for all  $e \in E$ . Hence  $(f, g) \in \Delta_{\text{CTF}}^\rho(G, \varepsilon)$ .

(b) Let  $(f, g)$  be a complementary  $(1, 1)$ -tension-flow of  $(G, \varepsilon)$ . Then  $(f, g) \in \Delta_{\text{CTF}}^\rho(G, \varepsilon)$  if and only if  $[\rho, \varepsilon](f + g)(e) > 0$  for  $e \in E$ ; i.e., if and only if  $[\rho, \varepsilon](f, g) \in \Delta_{\text{CTF}}^+(G, \rho)$ , since  $[\rho, \varepsilon](f, g) := ([\rho, \varepsilon]f, [\rho, \varepsilon]g)$  is a tension-flow of  $(G, \rho)$ ; and equivalently,  $(f, g) \in P_{\rho, \varepsilon} \Delta_{\text{CTF}}^+(G, \rho)$ , since  $P_{\rho, \varepsilon}(f, g) := [\rho, \varepsilon](f, g)$  and  $P_{\rho, \varepsilon}$  is an involution.

(c) Trivial.

(d) The bijection between  $\Delta_{\text{TN}}^+(G, B_\varepsilon)$  and  $\Delta_{\text{TN}}^+(G/C_\varepsilon, \varepsilon)$  is given by  $f \mapsto f|_{B_\varepsilon}$ ; and the bijection between  $\Delta_{\text{FL}}^+(G, C_\varepsilon)$  and  $\Delta_{\text{FL}}^+(G|C_\varepsilon, \varepsilon)$  is given by  $f \mapsto f|_{C_\varepsilon}$ .  $\square$

REMARK. The polytope  $\Delta_{\text{TN}}^+(G, B_\varepsilon)$  cannot be identified to the polytope  $\Delta_{\text{TN}}^+(G|B_\varepsilon, \varepsilon)$  because a tension of the digraph  $(G|B_\varepsilon, \varepsilon)$  can not be viewed as a tension of  $(G, \varepsilon)$  which vanishes on the edge subset of  $C_\varepsilon$ .

**Proposition 3.3.** (a) The counting function  $\kappa_\varepsilon(G; p, q)$  ( $\bar{\kappa}_\varepsilon(G; p, q)$ ) is a polynomial function of positive (nonnegative) integers  $p, q$  of degree  $r(G/C_\varepsilon) + n(G|C_\varepsilon)$ .

(b) Product Decomposition:

$$\kappa_\varepsilon(G; x, y) = \tau_\varepsilon(G/C_\varepsilon, x) \varphi_\varepsilon(G|C_\varepsilon, y), \quad (3.7)$$

$$\bar{\kappa}_\varepsilon(G; x, y) = \bar{\tau}_\varepsilon(G/C_\varepsilon, x) \bar{\varphi}_\varepsilon(G|C_\varepsilon, y). \quad (3.8)$$

Moreover,  $\tau_\varepsilon(G/C_\varepsilon, x)$ ,  $\varphi_\varepsilon(G|C_\varepsilon, y)$ ,  $\bar{\tau}_\varepsilon(G/C_\varepsilon, x)$ , and  $\bar{\varphi}_\varepsilon(G|C_\varepsilon, y)$  are the Ehrhart polynomials of the lattice polytopes  $\Delta_{\text{TN}}^+(G/C_\varepsilon, \varepsilon)$ ,  $\Delta_{\text{FL}}^+(G|C_\varepsilon, \varepsilon)$ ,  $\bar{\Delta}_{\text{TN}}^+(G/C_\varepsilon, \varepsilon)$ , and  $\bar{\Delta}_{\text{FL}}^+(G|C_\varepsilon, \varepsilon)$  respectively.

(c) Reciprocity Law:

$$\kappa_\varepsilon(G; -x, -y) = (-1)^{r(G) + |E(C_\varepsilon)|} \bar{\kappa}_\varepsilon(G; x, y). \quad (3.9)$$

(d) Specializations:

$$\kappa_\varepsilon(G; x, 1) = \tau_\varepsilon(G; x), \quad \kappa_\varepsilon(G; 1, y) = \varphi_\varepsilon(G, y); \quad (3.10)$$

$$\bar{\kappa}_\varepsilon(G; x, -1) = (-1)^{|E(C_\varepsilon)|} \bar{\tau}_\varepsilon(G, x), \quad (3.11)$$

$$\bar{\kappa}_\varepsilon(G; -1, y) = (-1)^{|E(B_\varepsilon)|} \bar{\varphi}_\varepsilon(G, y). \quad (3.12)$$

PROOF. (a) It is an immediate consequence of (b).

(b) It follows from the product decomposition (3.6) and the identifications of the lattice polytopes in Lemma 3.2(d).

(c) It follows from the Reciprocity Law (2.2) of Ehrhart polynomials and the relation

$$r(G/C_\varepsilon) + n(G|C_\varepsilon) = |E(C_\varepsilon)| + 2c(G|C_\varepsilon) - r(G) - 2c(G).$$

(d) Let  $p, q$  be positive integers. Let  $(f, g)$  be a positive integer-valued  $(p, 1)$ -tension-flow of  $(G, \varepsilon)$ . Since  $0 < g(e) < 1$  is impossible for any  $e \in E$ , we must have  $E(C_\varepsilon) = \emptyset$  and  $E(B_\varepsilon) = E$ ; i.e., the orientation  $\varepsilon$  must be acyclic. So  $(f, g)$  is reduced to a positive integer-valued  $p$ -tension  $f$  of  $(G, \varepsilon)$ . We thus have  $\kappa_\varepsilon(G; p, 1) = \tau_\varepsilon(G, p)$ . Note that  $\tau_\varepsilon$  is not the zero polynomial if and only if  $G$  is loopless and contains some edges.

Analogously, let  $(f, g)$  be a positive integer-valued  $(1, q)$ -tension-flow of  $(G, \varepsilon)$ . Since  $0 < f(e) < 1$  is impossible for any  $e \in E$ , we must have  $E(B_\varepsilon) = \emptyset$  and  $E(C_\varepsilon) = E$ ; i.e., the orientation  $\varepsilon$  must be totally cyclic. So  $(f, g)$  is reduced to a positive integer-valued  $q$ -flow  $g$  of  $(G, \varepsilon)$ . We thus have  $\kappa_\varepsilon(G; 1, q) = \varphi_\varepsilon(G, q)$ . Note that  $\varphi_\varepsilon$  is not the zero polynomial if and only if  $G$  is bridgeless and contains some edges. We finish the proof of (3.10).

Now applying the Reciprocity Law (3.9) and the formula (3.10), we have

$$\begin{aligned} \bar{\kappa}_\varepsilon(G; x, -1) &= (-1)^{r(G)+|E(C_\varepsilon)|} \kappa_\varepsilon(G; -x, 1) \\ &= (-1)^{r(G)+|E(C_\varepsilon)|} \tau_\varepsilon(G, -x) \\ &= (-1)^{|E(C_\varepsilon)|} \bar{\tau}_\varepsilon(G, x); \\ \bar{\kappa}_\varepsilon(G; -1, y) &= (-1)^{r(G)+|E(C_\varepsilon)|} \kappa_\varepsilon(G; 1, -y) \\ &= (-1)^{r(G)+|E(C_\varepsilon)|} \varphi_\varepsilon(G, -y) \\ &= (-1)^{|E(B_\varepsilon)|} \bar{\varphi}_\varepsilon(G, y). \end{aligned}$$

The last two equality in both follow from the Reciprocity Laws

$$\tau_\varepsilon(G, -x) = (-1)^{r(G)} \bar{\tau}_\varepsilon(G, x), \quad \varphi_\varepsilon(G, -y) = (-1)^{n(G)} \bar{\varphi}_\varepsilon(G, y) \quad (3.13)$$

respectively; see (1.9) of Theorem 1.2 in [12] and (1.7) of Theorem 1.1 in [13]. We finish the proof of (3.11) and (3.12).  $\square$

#### PROOF OF THEOREM 1.1

(a) Being a polynomial follows from (b) and (c) of the same theorem, and from (a) and (b) of Proposition 3.3. The independence of the chosen orientation  $\varepsilon$  follows from the bijection  $P_{\rho, \varepsilon} : (p, q)\Delta_{\text{CTF}}(G, \varepsilon) \rightarrow (p, q)\Delta_{\text{CTF}}(G, \rho)$  which sends lattice points to lattice points.

(b) The decomposition (1.11) follows from the disjoint composition of the complementary polyhedron  $\Delta_{\text{CTF}}(G, \varepsilon)$  (see Lemma 3.2(a)) and the property that the involution  $P_{\rho, \varepsilon}$  sends lattice points to lattice points bijectively (see Lemma 3.2(b)). The decomposition (1.12) is trivial by definition of  $\bar{\kappa}_\varepsilon$ .

(c) It follows from the Reciprocity Law (3.9).

(d) Let  $p, q$  be integers larger than or equal to 2. Consider an integer-valued complementary  $(p, 1)$ -tension-flow  $(f, g)$  of  $(G, \varepsilon)$ . Since  $|g| < 1$  implies  $g = 0$ , then  $f$  is a nowhere-zero  $p$ -tension of  $(G, \varepsilon)$ , and  $(f, 0)$  is identified to the nowhere-zero  $p$ -tension  $f$  of  $(G, \varepsilon)$ . Thus  $\kappa_\varepsilon(G; p, 1) = \tau_\varepsilon(G, p)$ . Likewise, an integer-valued complementary  $(1, q)$ -tension-flow  $(f, g)$  of  $(G, \varepsilon)$  implies that  $f = 0$  and  $g$  is a nowhere-zero  $q$ -flow of  $(G, \varepsilon)$ . Hence  $\kappa_\varepsilon(G; 1, q) = \varphi_\varepsilon(G, q)$ . We finished proof of (1.15).

Now consider the case of  $y = 1$  and the case of  $x = 1$  in (1.11). Applying Proposition 3.3(d), (1.11) becomes

$$\tau_{\mathbb{Z}}(G, x) = \sum_{\rho \in \mathcal{O}_{\text{AC}}(G)} \tau_{\rho}(G, x), \quad (3.14)$$

$$\varphi_{\mathbb{Z}}(G, y) = \sum_{\rho \in \mathcal{O}_{\text{TC}}(G)} \varphi_{\rho}(G, y). \quad (3.15)$$

We have recovered (1.7) of Theorem 1.2 in [12] (p.428) and (1.8) of Theorem 1.1(b) in [13]; see also [20, 21] for equivalent versions. The dual polynomials  $\bar{\tau}_{\mathbb{Z}}$  and  $\bar{\varphi}_{\mathbb{Z}}$  have similar decompositions by their definitions:

$$\bar{\tau}_{\mathbb{Z}}(G, x) = \sum_{\rho \in \mathcal{O}_{\text{AC}}(G)} \bar{\tau}_{\rho}(G, x), \quad (3.16)$$

$$\bar{\varphi}_{\mathbb{Z}}(G, y) = \sum_{\rho \in \mathcal{O}_{\text{TC}}(G)} \bar{\varphi}_{\rho}(G, y). \quad (3.17)$$

Analogously, consider the case of  $y = -1$  and the case of  $x = -1$  in (1.12). We have

$$\begin{aligned} \bar{\kappa}_{\mathbb{Z}}(G; x, -1) &= \sum_{\rho \in \mathcal{O}(G)} (-1)^{r(G) + |E(C_{\rho})|} \kappa_{\rho}(G; -x, 1) \\ &= \sum_{\rho \in \mathcal{O}_{\text{AC}}(G)} (-1)^{r(G)} \tau_{\rho}(G, -x) \\ &= \sum_{\rho \in \mathcal{O}_{\text{AC}}(G)} \bar{\tau}_{\rho}(G, x) = \bar{\tau}_{\mathbb{Z}}(G, x), \end{aligned}$$

where the first equality follows from (1.14), the second from (3.10) and that  $\tau_{\rho} \neq 0$  if and only if  $\rho \in \mathcal{O}_{\text{AC}}(G)$ , and the third from (3.16). Using  $r(G) + n(G) = |E|$  and  $|E| = |E(B_{\rho})| + |E(C_{\rho})|$ , a similar argument implies

$$\begin{aligned} \bar{\kappa}_{\mathbb{Z}}(G; -1, y) &= \sum_{\rho \in \mathcal{O}(G)} (-1)^{n(G) + |E(B_{\rho})|} \kappa_{\rho}(G; 1, -y) \\ &= \sum_{\rho \in \mathcal{O}_{\text{TC}}(G)} (-1)^{n(G)} \varphi_{\rho}(G, -y) \\ &= \sum_{\rho \in \mathcal{O}_{\text{TC}}(G)} \bar{\varphi}_{\rho}(G, y) = \bar{\varphi}_{\mathbb{Z}}(G, y), \end{aligned}$$

where the first equality follows from (1.14), the second equality follows from (3.10) and that  $\varphi_{\rho} \neq 0$  if and only if  $\rho \in \mathcal{O}_{\text{TC}}(G)$ , and the last equality follows from (3.17). We finish the proof of (1.16).

(e) Finally, let  $X \subseteq E$  be an edge subset. For each orientation  $\rho \in \mathcal{O}(G)$  such that  $E(C_{\rho}) = X$  (such an orientation may not exist), let  $\rho|_X$  denote the restriction of  $\rho$  on  $G|X$ , and let  $\rho_{/X}$  denote the induced orientation on  $G/X$ . Note that  $\rho|_X$

is totally cyclic on  $G|X$  and  $\rho|_X$  is acyclic on  $G/X$ . Then

$$\begin{aligned}\kappa_{\mathbb{Z}}(G; x, y) &= \sum_{X \subseteq E} \sum_{\substack{\rho \in \mathcal{O}(G) \\ C_\rho = X}} \tau_\rho(G/C_\rho, x) \varphi_\rho(G|C_\rho, y) \\ &= \sum_{X \subseteq E} \sum_{\substack{\rho \in \mathcal{O}(G) \\ C_\rho = X}} \tau_{\rho|_X}(G/X, x) \varphi_{\rho|_X}(G|X, y) \\ &= \sum_{X \subseteq E} \sum_{\substack{\rho \in \mathcal{O}_{\text{AC}}(G/X) \\ \sigma \in \mathcal{O}_{\text{TC}}(G|X)}} \tau_\rho(G/X, x) \varphi_\sigma(G|X, y),\end{aligned}$$

where the first equality follows from the equations (1.11) and (3.7). Apply (3.14) to the graph  $G/X$  and (3.15) to the graph  $G|X$  in the above last equality; we obtain (1.17) immediately. Analogously, the equations (1.12) and (3.8) imply that

$$\begin{aligned}\bar{\kappa}_{\mathbb{Z}}(G; x, y) &= \sum_{X \subseteq E} \sum_{\substack{\rho \in \mathcal{O}(G) \\ C_\rho = X}} \bar{\tau}_\rho(G/C_\rho, x) \bar{\varphi}_\rho(G|C_\rho, y) \\ &= \sum_{X \subseteq E} \sum_{\substack{\rho \in \mathcal{O}(G) \\ C_\rho = X}} \bar{\tau}_{\rho|_X}(G/X, x) \bar{\varphi}_{\rho|_X}(G|X, y) \\ &= \sum_{X \subseteq E} \sum_{\substack{\rho \in \mathcal{O}_{\text{AC}}(G/X) \\ \sigma \in \mathcal{O}_{\text{TC}}(G|X)}} \bar{\tau}_\rho(G/X, x) \bar{\varphi}_\sigma(G|X, y),\end{aligned}$$

Apply (3.16) to  $G/X$  and (3.17) to  $G|X$ . We obtain (1.18).  $\square$

The polynomials  $\kappa_{\mathbb{Z}}$  and  $\bar{\kappa}_{\mathbb{Z}}$  have the following particular combinatorial interpretations at some special integers.

**Corollary 3.4.** (a)  $\bar{\kappa}_{\mathbb{Z}}(G; 0, 0) = |\mathcal{O}(G)|$ ,

$$\begin{aligned}|\kappa_{\mathbb{Z}}(G; 1, 0)| &= \bar{\kappa}_{\mathbb{Z}}(G; -1, 0) = |\mathcal{O}_{\text{TC}}(G)|, \\ |\kappa_{\mathbb{Z}}(G; 0, 1)| &= \bar{\kappa}_{\mathbb{Z}}(G; 0, -1) = |\mathcal{O}_{\text{AC}}(G)|, \\ \kappa_{\mathbb{Z}}(G; 1, 1) &= \bar{\kappa}_{\mathbb{Z}}(G; -1, -1) = 0.\end{aligned}$$

(b) Let  $\mathcal{O}_{\text{CU}}(G)$ ,  $\mathcal{O}_{\text{EU}}(G)$ , and  $\mathcal{O}_{\text{CE}}(G)$  be the sets of orientations  $\rho$  such that  $(G, \rho)$  is a locally directed cut, a directed Eulerian graph, and a disjoint union of a locally directed cut and a directed Eulerian subgraph, respectively. Then

$$\begin{aligned}\kappa_{\mathbb{Z}}(G; 2, 1) &= |\bar{\kappa}_{\mathbb{Z}}(G; -2, -1)| = |\mathcal{O}_{\text{CU}}(G)|, \\ \kappa_{\mathbb{Z}}(G; 1, 2) &= |\bar{\kappa}_{\mathbb{Z}}(G; -1, -2)| = |\mathcal{O}_{\text{EU}}(G)|, \\ \kappa_{\mathbb{Z}}(G; 2, 2) &= |\mathcal{O}_{\text{CE}}(G)|.\end{aligned}$$

(c) For each orientation  $\rho$ , let  $[\rho]_{\text{CU}}$ ,  $[\rho]_{\text{EU}}$ , and  $[\rho]_{\text{CE}}$  denote the equivalence classes of  $\mathcal{O}(G)$  under the cut, Eulerian, and cut-Eulerian equivalence relations

respectively. Then

$$\begin{aligned}\bar{\kappa}_{\mathbb{Z}}(G; 1, 0) &= \sum_{\rho \in \mathcal{O}(G)} \#[\rho]_{\text{CU}}, \\ \bar{\kappa}_{\mathbb{Z}}(G; 0, 1) &= \sum_{\rho \in \mathcal{O}(G)} \#[\rho]_{\text{EU}}, \\ \bar{\kappa}_{\mathbb{Z}}(G; 1, 1) &= \sum_{\rho \in \mathcal{O}(G)} \#[\rho]_{\text{CE}}.\end{aligned}$$

PROOF. (a) Since  $\bar{\kappa}_{\rho}(G; 0, 0) = 1$  for  $\rho \in \mathcal{O}(G)$ , then by (1.12) we have  $\bar{\kappa}_{\mathbb{Z}}(G; 0, 0) = |\mathcal{O}(G)|$ .

Since  $\tau_{\rho}(G, 0) = (-1)^{r(G)}$  for  $\rho \in \mathcal{O}_{\text{AC}}(G)$ , then by (3.14) we have  $\tau_{\mathbb{Z}}(G, 0) = (-1)^{r(G)}|\mathcal{O}_{\text{AC}}(G)|$ . Since  $\varphi_{\rho}(G, 0) = (-1)^{n(G)}$  for  $\rho \in \mathcal{O}_{\text{TC}}(G)$ , then by (3.15) we have  $\varphi_{\mathbb{Z}}(G, 0) = (-1)^{n(G)}|\mathcal{O}_{\text{TC}}(G)|$ . According to Theorem 1.1(d), we see that  $\kappa_{\mathbb{Z}}(G; 1, 0) = (-1)^{n(G)}|\mathcal{O}_{\text{TC}}(G)|$  and  $\kappa_{\mathbb{Z}}(G; 0, 1) = (-1)^{r(G)}|\mathcal{O}_{\text{AC}}(G)|$ . Notice the Reciprocity Laws on  $\tau_{\mathbb{Z}}, \bar{\tau}_{\mathbb{Z}}$  and on  $\varphi_{\mathbb{Z}}, \bar{\varphi}_{\mathbb{Z}}$ . Again according to Theorem 1.1(d), we see that  $\bar{\kappa}_{\mathbb{Z}}(G; -1, 0) = |\mathcal{O}_{\text{TC}}(G)|$  and  $\bar{\kappa}_{\mathbb{Z}}(G; 0, -1) = |\mathcal{O}_{\text{AC}}(G)|$ .

Since  $\kappa_{\rho}(G; 1, 1) = 0$  for  $\rho \in \mathcal{O}(G)$ , then  $\bar{\kappa}_{\rho}(G; -1, -1) = 0$  for  $\rho \in \mathcal{O}(G)$  by (3.9). Hence by (1.11) and (1.12), we have  $\kappa_{\mathbb{Z}}(G; 1, 1) = \bar{\kappa}_{\mathbb{Z}}(G; -1, -1) = 0$ .

(b) If  $f$  is an integer-valued tension of a digraph  $(G, \rho)$  such that  $0 < f(e) < 2$  for all  $e \in E$ , then  $f \equiv 1$ . This means that  $(G, \rho)$  is a disjoint union of directed bonds, namely, a locally directed cut. Thus  $\tau_{\mathbb{Z}}(G, 2) = |\mathcal{O}_{\text{CU}}(G)|$  by (1.11), and subsequently,  $\bar{\tau}_{\mathbb{Z}}(G, -2) = (-1)^{r(G)}|\mathcal{O}_{\text{CU}}(G)|$ . According to Theorem 1.1(d), we have  $\kappa_{\mathbb{Z}}(G; 2, 1) = |\mathcal{O}_{\text{CU}}(G)|$  and  $\bar{\kappa}_{\mathbb{Z}}(G; -2, -1) = (-1)^{r(G)}|\mathcal{O}_{\text{CU}}(G)|$ .

If  $g$  is an integer-valued flow of a digraph  $(G, \rho)$  such that  $0 < g(e) < 2$  for  $e \in E$ , then  $g \equiv 1$ . This means that  $(G, \rho)$  is a disjoint union of directed circuits, namely, a directed Eulerian graph. Thus  $\varphi_{\mathbb{Z}}(G, 2) = |\mathcal{O}_{\text{EU}}(G)|$  by (1.11), and subsequently,  $\bar{\varphi}_{\mathbb{Z}}(G, -2) = (-1)^{n(G)}|\mathcal{O}_{\text{EU}}(G)|$ . According to Theorem 1.1(d), we have  $\kappa_{\mathbb{Z}}(G; 1, 2) = |\mathcal{O}_{\text{EU}}(G)|$  and  $\bar{\kappa}_{\mathbb{Z}}(G; -1, -2) = (-1)^{n(G)}|\mathcal{O}_{\text{EU}}(G)|$ .

Let  $(f, g)$  be an integer-valued complementary tension-flow of  $(G, \rho)$  such that  $0 < f(e) < 2$ ,  $0 < g(e) < 2$  for  $e \in E$ . Then  $f(e) + g(e) = 1$  for all  $e \in E$ . This means that  $(G, \rho)$  is a disjoint union of a locally directed cut and a directed Eulerian subgraph. Hence  $\kappa_{\mathbb{Z}}(G; 2, 2) = |\mathcal{O}_{\text{CE}}(G)|$  by (1.11).

(c) Note that  $\bar{\kappa}_{\rho}(G; 1, 0) = \bar{\tau}_{\rho}(G, 1)$  and  $\bar{\kappa}_{\rho}(G; 0, 1) = \bar{\varphi}_{\rho}(G, 1)$ . We then have  $\bar{\kappa}_{\rho}(G; 1, 0) = \#[\rho]_{\text{CU}}$  by Proposition 6.8 in [12],  $\bar{\kappa}_{\rho}(G; 0, 1) = \#[\rho]_{\text{EU}}$  by Proposition 5.5 in [13], and  $\kappa_{\rho}(G; 1, 1) = \#[\rho]_{\text{CE}}$  by Proposition 4.6. Now the desired formulas follow immediately from Equation (1.12).  $\square$

#### 4. Modular complementary polynomials

Let  $A, B$  be abelian groups of orders  $|A| = p, |B| = q$ . We define the counting function

$$\kappa(G; p, q) := |K(G, \varepsilon; A, B)|, \quad (4.1)$$

which turns out to be a polynomial function of positive integers  $p, q$ , called the *modular complementary polynomial* of  $G$ . To find the relationship between  $\kappa$  and  $\kappa_{\mathbb{Z}}$ , we need an equivalence relation on  $\mathcal{O}(G)$ . Two orientations  $\rho, \sigma$  on  $G$  are said to be *cut-Eulerian equivalent*, denoted  $\rho \sim_{\text{CE}} \sigma$ , if the subgraph induced by the edge

subset

$$E(\rho \neq \sigma) := \{e \in E \mid \rho(v, e) \neq \sigma(v, e), v \text{ is an end-vertex of } e\}$$

is a disjoint union of a locally directed cut and a directed Eulerian subgraph of both digraphs  $(G, \rho)$  and  $(G, \sigma)$ .

Let  $[\mathcal{O}(G)]$  denote the set of cut-Eulerian equivalence classes of  $\mathcal{O}(G)$ , and let  $\text{Rep}[\mathcal{O}(G)]$  be a set of distinct representatives of cut-Eulerian equivalence classes of  $\mathcal{O}(G)$ . For nonnegative integers  $p, q$ , recall the counting function

$$\bar{\kappa}(G; p, q) := \#\{(\rho, f, g) \mid \rho \in \text{Rep}[\mathcal{O}(G)], (f, g) \text{ is an integer-valued tension-flow of } (G, \rho) \text{ s.t. } 0 \leq f(e) \leq p, 0 \leq g(e) \leq q, e \in E\}. \quad (4.2)$$

It is clear that  $\bar{\kappa}$  is a polynomial function of nonnegative integers  $p, q$ , and is independent of the chosen set  $\text{Rep}[\mathcal{O}(G)]$  of distinct representatives, called the *dual modular complementary polynomial* of  $G$ . We introduce the following topological sum

$$[\tilde{\Delta}_{\text{CTF}}(G, \varepsilon)] := \sum_{[\rho] \in [\mathcal{O}(G)]} \bar{\Delta}_{\text{CTF}}^\rho(G, \varepsilon),$$

defined as a disjoint union of copies of  $\bar{\Delta}_{\text{CTF}}^\rho(G, \rho)$ , one copy for each equivalence class  $[\rho] \in [\mathcal{O}(G)]$ . Then  $\bar{\kappa}(G; p, q)$  counts the number of lattice points of  $(p, q)[\tilde{\Delta}_{\text{CTF}}(G, \varepsilon)]$ .

Let  $p, q$  be positive integers. Set  $\mathbb{R}_p := \mathbb{R}/p\mathbb{Z}$  and  $\mathbb{R}_q := \mathbb{R}/q\mathbb{Z}$ . There is an obvious homomorphism  $\text{Mod}_{p,q} : (\mathbb{R} \times \mathbb{R})^E \rightarrow (\mathbb{R}_p \times \mathbb{R}_q)^E$ , defined for  $(f, g) \in (\mathbb{R} \times \mathbb{R})^E$  by

$$\text{Mod}_{p,q}(f, g)(e) = (f(e) \bmod p, g(e) \bmod q), \quad e \in E. \quad (4.3)$$

For two orientations  $\rho, \sigma \in \mathcal{O}(G)$  and an edge subset  $S \subseteq E$ , there is an involution  $Q_{\rho, \sigma, S}^p : [0, p]^E \rightarrow [0, p]^E$  defined by

$$(Q_{\rho, \sigma, S}^p f)(e) := \begin{cases} p - f(e) & \text{if } e \in S, \rho(v, e) \neq \sigma(v, e), \\ f(e) & \text{otherwise,} \end{cases}$$

where  $f \in [0, p]^E$ ,  $e \in E$ , and  $v$  is an end-vertex of  $e$ . To find the relationship holding among the polytopes  $p\bar{\Delta}_{\text{TN}}(G, \rho) \times q\bar{\Delta}_{\text{FL}}(G, \rho)$  with  $\rho \in \mathcal{O}(G)$ , we need the involution  $Q_{\rho, \sigma}^{p,q} : ([0, p] \times [0, q])^E \rightarrow ([0, p] \times [0, q])^E$ , defined for  $(f, g) \in ([0, p] \times [0, q])^E$  by

$$Q_{\rho, \sigma}^{p,q}(f, g) = (Q_{\rho, \sigma, B_\sigma}^p f, Q_{\rho, \sigma, C_\sigma}^q g).$$

We establish the following lemmas.

**Lemma 4.1.** *Two orientations  $\rho, \sigma \in \mathcal{O}(G)$  are cut-Eulerian equivalent if and only if*

- (a)  $E(B_\rho) = E(B_\sigma)$ ,  $E(C_\rho) = E(C_\sigma)$ ;
- (b)  $E(B_\rho) \cap E(\rho \neq \sigma)$  is a locally directed cut of  $(G, \rho)$ ;
- (c)  $E(C_\rho) \cap E(\rho \neq \sigma)$  is a directed Eulerian subgraph of  $(G, \rho)$ .

**PROOF.** The sufficiency is trivial because  $E = E(B_\rho) \sqcup E(C_\rho)$  (see Lemma 3.1). For necessity, let us write  $E(\rho \neq \sigma) = B \sqcup C$ , where  $B$  is a locally directed cut and  $C$  a directed Eulerian subgraph of  $(G, \rho)$ . Clearly,  $C \subseteq E(C_\rho)$ . Since each directed circuit of  $C_\rho$  is disjoint from any directed bond contained in the locally directed cut  $(B, \rho)$  of  $(G, \rho)$ , then  $B \cap E(C_\rho) = \emptyset$ , and subsequently,  $B \subseteq E(B_\rho)$ . Thus we must have  $B = E(B_\rho) \cap E(\rho \neq \sigma)$  and  $C = E(C_\rho) \cap E(\rho \neq \sigma)$ . Now it follows that (b) and (c) are automatically true.

To prove (a), let  $\rho'$  be an orientation on  $G$  obtained from  $\rho$  by reversing the orientations on the edges of  $C$ . Then  $B = E(\rho' \neq \sigma)$  and  $E - B = E(\rho' = \sigma)$ . Since reversing the orientations on any strongly connected subdigraph does not change the strong connectedness, we have  $E(C_{\rho'}) = E(C_\rho)$ . Note that  $C_{\rho'}$  and  $C_\sigma$  are edge-disjoint from  $B$ . This means that  $C_{\rho'}$  and  $C_\sigma$  are contained in  $E(\rho' = \sigma)$ . Hence  $C_{\rho'} = C_\sigma$ . Therefore  $E(C_\rho) = E(C_\sigma)$ , and subsequently,  $E(B_\rho) = E(B_\sigma)$ .  $\square$

**Lemma 4.2.** (a) *The relation  $\sim_{\text{CE}}$  is an equivalence relation on  $\mathcal{O}(G)$ .*

(b) *Let  $\rho, \sigma \in \mathcal{O}(G)$  and  $\rho \sim_{\text{CE}} \sigma$ . If  $B_\rho$  is a locally directed cut, so is  $B_\sigma$ . If  $C_\rho$  is a directed Eulerian graph, so is  $C_\sigma$ .*

(c) *Let  $\rho, \sigma \in \mathcal{O}(G)$  and  $\rho \sim_{\text{CE}} \sigma$ . Then the following restriction*

$$Q_{\rho, \sigma}^{p, q} : p\bar{\Delta}_{\text{TN}}(G, B_\sigma) \times q\bar{\Delta}_{\text{FL}}(G, C_\sigma) \rightarrow p\bar{\Delta}_{\text{TN}}(G, B_\rho) \times q\bar{\Delta}_{\text{FL}}(G, C_\rho)$$

*is a bijection, sending lattice points to lattice points bijectively.*

(d)  *$\kappa_\rho(G; x, y) = \kappa_\sigma(G; x, y)$  if  $\rho \sim_{\text{CE}} \sigma$ .*

PROOF. (a) Let  $\varepsilon_i \in \mathcal{O}(G)$  ( $i = 1, 2, 3$ ) be orientations such that  $\varepsilon_1 \sim_{\text{CE}} \varepsilon_2$  and  $\varepsilon_2 \sim_{\text{CE}} \varepsilon_3$ . Set  $B := E(B_{\varepsilon_i})$  and  $C := E(C_{\varepsilon_i})$  by Lemma 4.1. Then  $\varepsilon_1 \sim_{\text{C}} \varepsilon_2$ ,  $\varepsilon_2 \sim_{\text{C}} \varepsilon_3$  on  $B$ , and  $\varepsilon_1 \sim_{\text{E}} \varepsilon_2$ ,  $\varepsilon_2 \sim_{\text{E}} \varepsilon_3$  on  $C$ . Thus  $\varepsilon_1 \sim_{\text{C}} \varepsilon_3$  on  $B$  by Lemma 6.5(a) of [12], and  $\varepsilon_1 \sim_{\text{C}} \varepsilon_3$  on  $C$  by Lemma 5.1(a) of [13]. Hence  $\varepsilon_1 \sim_{\text{CE}} \varepsilon_3$ .

(b) It follows from the fact that the subdigraph  $B_\rho|E(\rho = \sigma)$  is a directed cut and that  $C_\rho|E(\rho = \sigma)$  is a directed Eulerian subgraph of both  $(G, \rho)$  and  $(G, \sigma)$ .

(c) Let  $E(\rho \neq \sigma) = B' \sqcup C'$ , where  $B'$  is a locally directed cut and  $C'$  a directed Eulerian subgraph with the orientations both  $\rho$  and  $\sigma$ . Let  $f \in p\bar{\Delta}_{\text{TN}}(G, \sigma)$  and  $g \in q\bar{\Delta}_{\text{FL}}(G, \sigma)$ . We need to show that  $Q_{\rho, \sigma, B_\sigma}^p f \in p\bar{\Delta}_{\text{TN}}(G, \rho)$  and  $Q_{\rho, \sigma, C_\sigma}^q g \in q\bar{\Delta}_{\text{FL}}(G, \rho)$ . In fact, given an arbitrary directed circuit  $(C, \varepsilon_C)$  and directed bond  $(B, \varepsilon_B)$  of  $(G, \sigma)$ . We have

$$\sum_{e \in C} [\sigma, \varepsilon_C](e) f(e) = 0, \quad \sum_{e \in B} [\sigma, \varepsilon_B](e) g(e) = 0.$$

Then  $I := \sum_{e \in C} [\rho, \varepsilon_C](e) (Q_{\rho, \sigma, B_\sigma}^p f)(e)$  can be written as

$$\begin{aligned} I &= \sum_{e \in C \cap (C' \cup E(\rho = \sigma))} [\rho, \varepsilon_C](e) f(e) + \sum_{e \in C \cap B'} [\rho, \varepsilon_C](e) (p - f(e)) \\ &= \sum_{e \in C} [\sigma, \varepsilon_C](e) f(e) - p \sum_{e \in C \cap B'} [\sigma, \varepsilon_C](e) = 0. \end{aligned}$$

The second sum being zero follows from the fact that  $(B', \sigma)$  is a locally directed cut of  $(G, \sigma)$ . Hence  $Q_{\rho, \sigma, B_\sigma}^p f \in p\bar{\Delta}_{\text{TN}}(G, \rho)$ . Analogously,  $J := \sum_{e \in B} [\rho, \varepsilon_B](e) (Q_{\rho, \sigma, C_\sigma}^q g)(e)$  can be written as

$$\begin{aligned} J &= \sum_{e \in B \cap (B' \cup E(\rho = \sigma))} [\rho, \varepsilon_B](e) g(e) + \sum_{e \in B \cap C'} [\rho, \varepsilon_B](e) (q - g(e)) \\ &= \sum_{e \in B} [\sigma, \varepsilon_B](e) g(e) - q \sum_{e \in B \cap C'} [\sigma, \varepsilon_B](e) = 0. \end{aligned}$$

The second sum being zero follows from the fact that  $(C', \sigma)$  is a directed Eulerian subgraph of  $(G, \sigma)$ . Therefore  $Q_{\rho, \sigma, C_\sigma}^q g \in q\bar{\Delta}_{\text{FL}}(G, \rho)$ . It is clear that  $Q_{\rho, \sigma}^{p, q} = Q_{\rho, \sigma, B_\sigma}^p \times Q_{\rho, \sigma, C_\sigma}^q$  sends lattice points bijectively.

(d) It is a consequence of (c).  $\square$



For a real-valued function  $f : E \rightarrow \mathbb{R}$  and an orientation  $\rho$  on  $G$ , we associate an orientation  $\rho_f$  on  $G$ , defined for each  $(v, e) \in V \times E$  by

$$\rho_f(v, e) := \begin{cases} \rho(v, e) & \text{if } f(e) > 0, \\ -\rho(v, e) & \text{if } f(e) \leq 0. \end{cases} \quad (4.4)$$

Conversely, for orientations  $\rho, \sigma \in \mathcal{O}(G)$ , we associate a *symmetric difference function*  $I_{\rho, \sigma} : E \rightarrow \{0, 1\}$ , defined for each edge  $e \in E$  (at its one end-vertex  $v$ ) by

$$I_{\rho, \sigma}(e) := \begin{cases} 1 & \text{if } \rho(v, e) \neq \sigma(v, e), \\ 0 & \text{if } \rho(v, e) = \sigma(v, e). \end{cases} \quad (4.5)$$

**Lemma 4.3.** *Let  $(f_i, g_i) \in p\Delta_{\text{TN}}(G, B_\varepsilon) \times q\Delta_{\text{FL}}(G, C_\varepsilon)$ ,  $i = 1, 2$ . If*

$$(f_1, g_1) \equiv (f_2, g_2) \pmod{(p, q)},$$

*then  $\varepsilon_{f_1+g_1}$  and  $\varepsilon_{f_2+g_2}$  are cut-Eulerian equivalent.*

PROOF. Note that  $f_i(e) = 0$  for  $e \in E(C_\varepsilon)$ ,  $g_i(e) = 0$  for  $e \in E(B_\varepsilon)$ . Then  $\varepsilon_{f_i} = \varepsilon_{f_i+g_i}$  on  $E(B_\varepsilon)$ , and  $\varepsilon_{g_i} = \varepsilon_{f_i+g_i}$  on  $E(C_\varepsilon)$ . Since  $f_1 \equiv f_2 \pmod{p}$ , then  $\varepsilon_{f_1}$  and  $\varepsilon_{f_2}$  are cut-equivalent by Lemma 6.6 of [12]. Analogously, since  $g_1 \equiv g_2 \pmod{q}$ , then  $\varepsilon_{g_1}$  and  $\varepsilon_{g_2}$  are Eulerian-equivalent by Lemma 5.3 of [13]. Since  $E(\varepsilon_{f_1} \neq \varepsilon_{f_2}) \subseteq E(B_\varepsilon)$  and  $E(\varepsilon_{g_1} \neq \varepsilon_{g_2}) \subseteq E(C_\varepsilon)$ , it follows that the edge set  $D := E(\varepsilon_{f_1+g_1} \neq \varepsilon_{f_2+g_2})$  can be written as

$$\begin{aligned} D &= (D \cap B_\varepsilon) \sqcup (D \cap C_\varepsilon) \\ &= (E(\varepsilon_{f_1} \neq \varepsilon_{f_2}) \cap E(B_\varepsilon)) \sqcup (E(\varepsilon_{g_1} \neq \varepsilon_{g_2}) \cap E(C_\varepsilon)) \\ &= E(\varepsilon_{f_1} \neq \varepsilon_{f_2}) \sqcup E(\varepsilon_{g_1} \neq \varepsilon_{g_2}), \end{aligned}$$

which is a disjoint union of a locally directed cut and a directed Eulerian subgraph with the orientations  $\varepsilon_{f_i+g_i}$ . Hence  $\varepsilon_{f_1+g_1}$  and  $\varepsilon_{f_2+g_2}$  are cut-Eulerian equivalent.  $\square$

**Lemma 4.4.** *The restriction  $\text{Mod}_{p,q} : K_{\mathbb{Z}}(G, \varepsilon; p, q) \rightarrow K(G, \varepsilon; \mathbb{Z}_p, \mathbb{Z}_q)$  is well-defined and surjective.*

PROOF. It is clear that  $\text{Mod}_{p,q}$  is well-defined. Let  $(\tilde{f}, \tilde{g}) \in K(G, \varepsilon; \mathbb{Z}_p, \mathbb{Z}_q)$ , i.e.,  $\tilde{f} \in T(G, \varepsilon; \mathbb{Z}_p)$ ,  $\tilde{g} \in F(G, \varepsilon; \mathbb{Z}_q)$ , and  $\text{supp } \tilde{g} = \ker \tilde{f}$ . Then by Lemma 6.1 of [12], there exists an integer-valued  $p$ -tension  $f$  of  $(G, \varepsilon)$  such that  $\text{Mod}_p f = \tilde{f}$ , and by Lemma 5.3 of [13], there exists an integer-valued  $q$ -flow  $g$  of  $(G, \varepsilon)$  such that  $\text{Mod}_q g = \tilde{g}$ . Moreover,  $f(e) = 0$  if and only if  $\tilde{f}(e) = 0$ ;  $g(e) = 0$  if and only if  $\tilde{g}(e) = 0$ . Thus  $\ker f = \ker \tilde{f}$ ,  $\text{supp } g = \text{supp } \tilde{g}$ . Therefore  $\text{supp } g = \ker f$ . This means that  $(f, g) \in K_{\mathbb{Z}}(G, \varepsilon; p, q)$  and  $\text{Mod}_{p,q}(f, g) = (\tilde{f}, \tilde{g})$ . We have proved the surjectivity of the map  $\text{Mod}_{p,q}$ .  $\square$

**Lemma 4.5.** *Let  $\rho, \sigma, \omega \in \mathcal{O}(G)$  be cut-Eulerian equivalent orientations, and let  $(f, g) \in (p, q)\Delta_{\text{CTF}}^\rho(G, \varepsilon)$ . Then*

- (a)  $\varepsilon_{f+g} = \rho$ .
- (b)  $P_{\varepsilon, \sigma} Q_{\sigma, \rho}^{p, q} P_{\rho, \varepsilon}((p, q)\Delta_{\text{CTF}}^\rho(G, \varepsilon)) = (p, q)\Delta_{\text{CTF}}^\sigma(G, \varepsilon)$ .
- (c)  $P_{\varepsilon, \sigma} Q_{\sigma, \rho}^{p, q} P_{\rho, \varepsilon}(f, g) = P_{\varepsilon, \omega} Q_{\omega, \rho}^{p, q} P_{\rho, \varepsilon}(f, g)$  if and only if  $\sigma = \omega$ .
- (d)  $K(G, \varepsilon; p, q) \cap \text{Mod}_{p,q}^{-1}(\text{Mod}_{p,q}(f, g)) = \{P_{\varepsilon, \alpha} Q_{\alpha, \rho}^{p, q} P_{\rho, \varepsilon}(f, g) \mid \alpha \sim_{\text{CE}} \rho\}$ .

PROOF. (a) Note that  $[\rho, \varepsilon](e)(f+g)(e) > 0$  for all  $e \in E$ . Then for each edge  $e \in E$  and its one end-vertex  $v$ ,  $(f+g)(e) > 0$  if and only if  $\rho(v, e) = \varepsilon(v, e)$ . Since  $\varepsilon_{f+g}(v, e) = \varepsilon(v, e)$  if and only if  $(f+g)(e) > 0$ , we see that  $\varepsilon_{f+g} = \rho$ .

(b) Since  $P_{\rho, \varepsilon}$ ,  $P_{\varepsilon, \sigma}$ , and  $Q_{\sigma, \rho}^{p, q}$  are involutions, it follows that the composition  $P_{\varepsilon, \rho} Q_{\rho, \sigma, S}^{p, q} P_{\sigma, \varepsilon}$  is an involution. Note that

$$P_{\rho, \varepsilon}((p, q) \Delta_{\text{CTF}}^{\rho}(G, \varepsilon)) = (p, q) \Delta_{\text{CTF}}^{+}(G, \rho),$$

$$Q_{\sigma, \rho}^{p, q}((p, q) \Delta_{\text{CTF}}^{+}(G, \rho)) = (p, q) \Delta_{\text{CTF}}^{+}(G, \sigma),$$

and  $P_{\sigma, \varepsilon} = P_{\varepsilon, \sigma}$ . The desired identity follows.

(c) The equation  $P_{\varepsilon, \sigma} Q_{\sigma, \rho, S}^{p, q} P_{\rho, \varepsilon}(f, g) = P_{\varepsilon, \omega} Q_{\omega, \rho}^{p, q} P_{\rho, \varepsilon}(f, g)$  is equivalent to

$$P_{\varepsilon, \sigma} Q_{\sigma, \rho, B_{\rho}}^p P_{\rho, \varepsilon} f = P_{\varepsilon, \omega} Q_{\omega, \rho, B_{\rho}}^p P_{\rho, \varepsilon} f, \quad (4.6)$$

$$P_{\varepsilon, \sigma} Q_{\sigma, \rho, C_{\rho}}^q P_{\rho, \varepsilon} g = P_{\varepsilon, \omega} Q_{\omega, \rho, C_{\rho}}^q P_{\rho, \varepsilon} g. \quad (4.7)$$

It follows that (4.6) is equivalent to  $\rho = \sigma$  on  $B_{\rho}$  by Lemma 6.7(c) of [12], and that (4.7) is equivalent to  $\rho = \sigma$  on  $C_{\rho}$  by Lemma 5.4(c) of [13]. Hence (4.6) and (4.7) are equivalent to  $\rho = \sigma$  on  $E$ .

(d) By Lemma 6.7(d) of [12],  $T(G, B_{\varepsilon}; p) \cap \text{Mod}_p^{-1}(\text{Mod}_p f)$  consists of the tensions  $P_{\varepsilon, \alpha} Q_{\alpha, \rho, B_{\rho}}^p P_{\rho, \varepsilon} f$ , where  $\alpha \sim_{\text{C}} \rho$  on  $B_{\rho}$ . Similarly, by Lemma 5.4(d) of [13],  $F(G, C_{\varepsilon}; q) \cap \text{Mod}_q^{-1}(\text{Mod}_q g)$  consists of the flows  $P_{\varepsilon, \alpha} Q_{\alpha, \rho, C_{\rho}}^q P_{\rho, \varepsilon} g$ , where  $\alpha \sim_{\text{E}} \rho$  on  $C_{\rho}$ . Clearly,  $P_{\varepsilon, \alpha} Q_{\alpha, \rho, B_{\rho}}^p P_{\rho, \varepsilon} f$  is zero on  $C_{\rho}$ ;  $P_{\varepsilon, \alpha} Q_{\alpha, \rho, C_{\rho}}^q P_{\rho, \varepsilon} g$  is zero on  $B_{\rho}$ . It follows that  $K(G, \varepsilon; p, q) \cap \text{Mod}_{p, q}^{-1}(\text{Mod}_{p, q}(f, g))$  consists of the tension-flows  $P_{\varepsilon, \alpha} Q_{\alpha, \rho}^{p, q} P_{\rho, \varepsilon}(f, g)$ , where  $\alpha \sim_{\text{CE}} \rho$  on  $E$ .  $\square$

**Proposition 4.6.** *For each orientation  $\rho$ , let  $[\rho]_{\text{CU}}$ ,  $[\rho]_{\text{EU}}$ , and  $[\rho]_{\text{CE}}$  denote its equivalence classes under the cut equivalence, Eulerian equivalence, and cut-Eulerian equivalence relations on  $\mathcal{O}(G)$  respectively. Then*

$$\#[\rho]_{\text{CE}} = \#[\rho]_{\text{CU}} \cdot \#[\rho]_{\text{EU}} \quad (4.8)$$

*and equals the number of 0-1 complementary tension-flows of  $(G, \rho)$ , i.e.,*

$$\#[\rho]_{\text{CE}} = \bar{\kappa}_{\rho}(G; 1, 1) = |(\mathbb{Z}^2)^E \cap \bar{\Delta}_{\text{CTF}}^{+}(G, \rho)|. \quad (4.9)$$

PROOF. The map  $[\rho]_{\text{CU}} \times [\rho]_{\text{EU}} \rightarrow [\rho]_{\text{CE}}$  given by  $(\sigma, \omega) \mapsto (\sigma|_{B_{\rho}}) \cup (\omega|_{C_{\rho}})$  is clearly a bijection, where  $\sigma|_{B_{\rho}}$  is the restriction of the orientation  $\sigma$  to  $E(B_{\rho})$ , and  $\omega|_{C_{\rho}}$  the restriction of  $\omega$  to  $E(C_{\rho})$ . The equality (4.8) follows immediately.

Let  $\sigma$  be an orientation that is cut-Eulerian equivalent to  $\rho$ . Let  $I_{\sigma}$  denote the symmetric difference function  $I_{\sigma, \rho}$  defined by (4.5). Then  $I_{\sigma}|_{B_{\rho}}$  is a tension on  $(G/C_{\rho}, \rho)$  by Proposition 6.8 in [12], and can be extended to a tension  $I'_{\sigma}$  on  $(G, \rho)$  such that  $I'_{\sigma}|_{C_{\rho}} = 0$ . Likewise,  $I_{\sigma}|_{C_{\rho}}$  is a flow on  $(G|B_{\rho}, \rho)$  by Lemma 5.5 in [13], and can be extended to a flow  $I''_{\sigma}$  on  $(G, \rho)$  such that  $I''_{\sigma}|_{B_{\rho}} = 0$ . Hence  $I_{\sigma}$  is decomposed into a 0-1 tension-flow  $(I'_{\sigma}, I''_{\sigma})$  of  $(G, \rho)$ . We then have a well-defined map

$$[\rho]_{\text{CE}} := \{\sigma \in \mathcal{O}(G) \mid \sigma \sim_{\text{CE}} \rho\} \rightarrow (\mathbb{Z}^2)^E \cap \bar{\Delta}_{\text{CTF}}^{+}(G, \rho), \quad \sigma \mapsto (I'_{\sigma}, I''_{\sigma}).$$

The map is clearly injective.

For surjectivity of the map, let  $(f, g) \in \bar{\Delta}_{\text{CTF}}^{+}(G, \rho)$  be a 0-1 tension-flow. Set  $\sigma := -\rho_{f+g}$ , where  $\rho_{f+g}$  is the orientation associated with  $f+g$  and  $\rho$  by (4.4).

Then  $I_{\sigma,\rho} = f + g$ . Since

$$\begin{aligned} E(\sigma \neq \rho) &= \{e \in E \mid (f + g)(e) = 1\} \\ &= \{e \in E(B_\rho) \mid f(e) = 1\} \sqcup \{e \in E(C_\rho) \mid g(e) = 1\} \end{aligned}$$

is a disjoint union of a locally directed cut and a directed Eulerian subgraph with the orientation  $\rho$ , we see that  $\sigma \sim_{\text{CE}} \rho$ . Thus the map is surjective.  $\square$

**Lemma 4.7.** *For each orientation  $\rho$  on  $G$ , let  $[\rho] := \{\sigma \in \mathcal{O}(G) \mid \sigma \sim_{\text{CE}} \rho\}$  and  $\Delta_{\text{CTF}}^{[\rho]}(G, \varepsilon) := \bigsqcup_{\sigma \in [\rho]} \Delta_{\text{CTF}}^\sigma(G, \varepsilon)$ . Then*

$$\kappa_\rho(G; p, q) = \#\text{Mod}_{p,q}((\mathbb{Z}^2)^E \cap (p, q)\Delta_{\text{CTF}}^\rho(G, \varepsilon)) \quad (4.10)$$

$$= \#\text{Mod}_{p,q}((\mathbb{Z}^2)^E \cap (p, q)\Delta_{\text{CTF}}^{[\rho]}(G, \varepsilon)); \quad (4.11)$$

$$K(G, \varepsilon; \mathbb{Z}_p, \mathbb{Z}_q) = \bigsqcup_{[\rho] \in [\mathcal{O}(G)]} \text{Mod}_{p,q}((\mathbb{Z}^2)^E \cap (p, q)\Delta_{\text{CTF}}^\rho(G, \varepsilon))^E \quad (4.12)$$

$$= \bigsqcup_{[\rho] \in [\mathcal{O}(G)]} \text{Mod}_{p,q}((\mathbb{Z}^2)^E \cap (p, q)\Delta_{\text{CTF}}^{[\rho]}(G, \varepsilon))^E. \quad (4.13)$$

PROOF. Notice that for each tension-flow  $(f, g) \in (p, q)\Delta_{\text{CTF}}^\rho(G, \varepsilon)$ , we have

$$\begin{aligned} \#[\rho] &= \#\{P_{\varepsilon,\sigma}Q_{\sigma,\rho}^{p,q}P_{\rho,\varepsilon}(f, g) \mid \sigma \sim_{\text{CE}} \rho\} \\ &= |K(G, \varepsilon; p, q) \cap \text{Mod}_{p,q}^{-1}\text{Mod}_{p,q}(f, g)|; \end{aligned} \quad (4.14)$$

the first equality follows from Lemma 4.5(c) and the second from Lemma 4.5(d). Then

$$\begin{aligned} (p, q)\Delta_{\text{CTF}}^{[\rho]}(G, \varepsilon) &= \bigsqcup_{\sigma \in [\rho]} P_{\varepsilon,\sigma}Q_{\sigma,\rho}^{p,q}P_{\rho,\varepsilon}((p, q)\Delta_{\text{CTF}}^\rho(G, \varepsilon)) \\ &= \bigsqcup_{\substack{\sigma \in [\rho] \\ (f, g) \in (p, q)\Delta_{\text{CTF}}^\rho(G, \varepsilon)}} \{P_{\varepsilon,\sigma}Q_{\sigma,\rho}^{p,q}P_{\rho,\varepsilon}(f, g)\} \\ &= \bigsqcup_{(f, g) \in (p, q)\Delta_{\text{CTF}}^\rho(G, \varepsilon)} K(G, \varepsilon; p, q) \cap \text{Mod}_{p,q}^{-1}\text{Mod}_{p,q}(f, g) \\ &= K(G, \varepsilon; p, q) \cap \text{Mod}_{p,q}^{-1}\text{Mod}_{p,q}(p, q)\Delta_{\text{CTF}}^\rho(G, \varepsilon); \end{aligned} \quad (4.15)$$

the first equality follows from Lemma 4.5(b), the second is straightforward, the third follows from Lemma 4.5(d), and the last is trivial. Since the orientation  $\rho$  on the left-hand side of (4.15) can be replaced by any orientation  $\sigma \in [\rho]$ , we thus have

$$(p, q)\Delta_{\text{CTF}}^{[\rho]}(G, \varepsilon) = K(G, \varepsilon; p, q) \cap \text{Mod}_{p,q}^{-1}\text{Mod}_{p,q}(p, q)\Delta_{\text{CTF}}^{[\rho]}(G, \varepsilon). \quad (4.16)$$

Now on the one hand, by (4.14), (4.15), and (4.16) we have

$$|(\mathbb{Z}^2)^E \cap (p, q)\Delta_{\text{CTF}}^{[\rho]}(G, \varepsilon)| = |\text{Mod}_{p,q}((\mathbb{Z}^2)^E \cap (p, q)\Delta_{\text{CTF}}^\rho(G, \varepsilon))| \cdot \#[\rho] \quad (4.17)$$

$$= |\text{Mod}_{p,q}((\mathbb{Z}^2)^E \cap (p, q)\Delta_{\text{CTF}}^{[\rho]}(G, \varepsilon))| \cdot \#[\rho]. \quad (4.18)$$

On the other hand, recall that  $\kappa_\rho(G; p, q) = \kappa_\sigma(G; p, q)$  if  $\sigma \sim_{\text{CE}} \rho$  (see Lemma 4.2(d)); by definition of  $\Delta_{\text{CTF}}^{[\rho]}(G, \varepsilon)$  we trivially have

$$|(\mathbb{Z}^2)^E \cap (p, q)\Delta_{\text{CTF}}^{[\rho]}(G, \varepsilon)| = \kappa_\rho(G; p, q) \cdot \#[\rho]. \quad (4.19)$$

Equate (4.17), (4.18), and (4.19); we obtain (4.10) and (4.11).

Notice the disjoint decomposition in Lemma 3.2(a). Using the notation  $\Delta_{\text{CTF}}^{[\rho]}(G, \varepsilon)$  and applying (4.15), we have

$$\begin{aligned} (p, q)\Delta_{\text{CTF}}(G, \varepsilon) &= \bigsqcup_{[\rho] \in [\mathcal{O}(G)]} \bigsqcup_{\sigma \in [\rho]} (p, q)\Delta_{\text{CTF}}^{\sigma}(G, \varepsilon) \\ &= \bigsqcup_{[\rho] \in [\mathcal{O}(G)]} K(G, \varepsilon; p, q) \cap \text{Mod}_{p,q}^{-1} \text{Mod}_{p,q}(p, q)\Delta_{\text{CTF}}^{\rho}(G, \varepsilon) \\ &= \bigsqcup_{[\rho] \in [\mathcal{O}(G)]} K(G, \varepsilon; p, q) \cap \text{Mod}_{p,q}^{-1} \text{Mod}_{p,q}(p, q)\Delta_{\text{CTF}}^{[\rho]}(G, \varepsilon). \end{aligned}$$

Note that  $K(G, \varepsilon; p, q) = (p, q)\Delta_{\text{CTF}}(G, \varepsilon)$ . Apply the map  $\text{Mod}_{p,q}$  to both sides and restrict to the integral lattice  $(\mathbb{Z}^2)^E$ ; we obtain (4.12) and (4.13) by Lemma 4.4.  $\square$

#### PROOF OF THEOREM 1.2

(a) Being a polynomial follows from the decomposition formulas (1.19) and (1.20). The independence of the chosen  $\text{Rep}[\mathcal{O}(G)]$  follows from the fact that  $\kappa_{\rho}(G; x, y) = \kappa_{\sigma}(G; x, y)$  if  $\rho \sim_{\text{CE}} \sigma$ .

(b) Let  $p, q$  be positive integers. Taking the cardinality on both sides of (4.13), we have

$$\kappa(G; p, q) = \sum_{[\rho] \in [\mathcal{O}(G)]} |\text{Mod}_{p,q}((\mathbb{Z}^2)^E \cap (p, q)\Delta_{\text{CTF}}^{[\rho]}(G, \varepsilon))|.$$

Taking account of (4.11), we obtain (1.19). Equation (1.20) is followed by the definition of  $\bar{\kappa}$ .

(c) The Reciprocity Laws are trivial by the Reciprocity Law (3.9).

(d) Let  $A, B$  be finite abelian groups. If  $|A| = p, |B| = 1$ , i.e.,  $B = \{0\}$ , then a tension-flow  $(f, g) \in \Omega(G, \varepsilon; A, B)$  is complementary if and only if  $g \equiv 0$  and  $f$  is a nowhere-zero tension of  $(G, \varepsilon)$ . Thus  $\kappa(G; x, 1) = \tau(G, x)$ . If  $|A| = 1, |B| = q$ , i.e.,  $A = \{0\}$ , then a tension-flow  $(f, g) \in \Omega(G, \varepsilon; A, B)$  is complementary if and only if  $f \equiv 0$  and  $g$  is a nowhere-zero flow of  $(G, \varepsilon)$ . Thus  $\kappa(G; 1, y) = \varphi(G, y)$ . We have proved (1.23).

Since  $\kappa_{\rho}(G; x, 1) = \tau_{\rho}(G, x)$  and  $\kappa_{\rho}(G; 1, y) = \varphi_{\rho}(G, y)$ , the decomposition formulas (1.19) and (1.20) become the following

$$\tau(G, x) = \sum_{[\rho] \in [\mathcal{O}_{\text{ac}}(G)]} \tau_{\rho}(G, x), \quad (4.20)$$

$$\varphi(G, y) = \sum_{[\rho] \in [\mathcal{O}_{\text{rc}}(G)]} \varphi_{\rho}(G, y). \quad (4.21)$$

The dual polynomials  $\bar{\tau}$  and  $\bar{\varphi}$  have the similar decomposition formulas (by definitions):

$$\bar{\tau}(G, x) = \sum_{[\rho] \in [\mathcal{O}_{\text{ac}}(G)]} \bar{\tau}_{\rho}(G, x), \quad (4.22)$$

$$\bar{\varphi}(G, y) = \sum_{[\rho] \in [\mathcal{O}_{\text{rc}}(G)]} \bar{\varphi}_{\rho}(G, y). \quad (4.23)$$

Consider the case of  $y = -1$  and the case of  $x = -1$ ; we obtain (1.24) as follows:

$$\begin{aligned}
\bar{\kappa}(G; x, -1) &= \sum_{[\rho] \in [\mathcal{O}_{\text{ac}}(G)]} (-1)^{r(G) + |E(C_\rho)|} \kappa_\rho(G; -x, 1) \\
&= \sum_{[\rho] \in [\mathcal{O}_{\text{ac}}(G)]} (-1)^{r(G)} \tau_\rho(G, -x) \quad (\text{since } |C_\rho| = 0) \\
&= \sum_{[\rho] \in [\mathcal{O}_{\text{ac}}(G)]} \bar{\tau}_\rho(G, x) = \bar{\tau}(G, x), \\
\bar{\kappa}(G; -1, y) &= \sum_{[\rho] \in [\mathcal{O}_{\text{tc}}(G)]} (-1)^{n(G) + |E(B_\rho)|} \kappa_\rho(G; 1, -y) \\
&= \sum_{[\rho] \in [\mathcal{O}_{\text{tc}}(G)]} (-1)^{n(G)} \varphi_\rho(G, -y) \quad (\text{since } |B_\rho| = 0) \\
&= \sum_{[\rho] \in [\mathcal{O}_{\text{tc}}(G)]} \bar{\varphi}_\rho(G, y) = \bar{\varphi}(G, y),
\end{aligned}$$

where the first equality of both follows from (1.22), the second equality of both follows from (1.23), the third equality of both follows from (3.9) (with  $E(C_\rho) = \emptyset$  and  $E(B_\rho) = \emptyset$  respectively), the last equality in the former follows from (4.22), and the last equality of the latter follows from (4.23).

(e) For a subset  $X \subseteq E$  and an orientation  $\rho \in \mathcal{O}(G)$ , we identify the edge set  $E(G/X)$  as  $E - X$ , write the restriction of  $\rho$  to  $X$  as  $\rho|_X$ , and the induced orientation by  $\rho$  on  $G/X$  as  $\rho|_X$ . If  $X = E(C_\rho)$ , then  $\rho|_X$  is acyclic and  $\rho|_X$  is totally cyclic. Thus

$$\begin{aligned}
\kappa(G; x, y) &= \sum_{X \subseteq E} \sum_{\substack{\rho \in [\mathcal{O}(G)] \\ E(C_\rho) = X}} \tau_{\rho|_X}(G/X, x) \varphi_{\rho|_X}(G|X, y) \\
&= \sum_{X \subseteq E} \sum_{\substack{\rho \in [\mathcal{O}_{\text{ac}}(G/X)] \\ \sigma \in [\mathcal{O}_{\text{tc}}(G|X)]}} \tau_\rho(G/X, x) \varphi_\sigma(G|X, y) \\
&= \sum_{X \subseteq E} \tau(G/X, x) \varphi(G|X, y).
\end{aligned}$$

The last equality follows from the decomposition (4.20) to the graph  $G/X$  and the decomposition (4.21) to  $G|X$ . Analogously,

$$\begin{aligned}
\bar{\kappa}(G; x, y) &= \sum_{X \subseteq E} \sum_{\substack{\rho \in [\mathcal{O}(G)] \\ E(C_\rho) = X}} \bar{\tau}_{\rho|_X}(G/X, x) \bar{\varphi}_{\rho|_X}(G|X, y) \\
&= \sum_{X \subseteq E} \sum_{\substack{\rho \in [\mathcal{O}_{\text{ac}}(G/X)] \\ \sigma \in [\mathcal{O}_{\text{tc}}(G|X)]}} \bar{\tau}_\rho(G/X, x) \bar{\varphi}_\sigma(G|X, y) \\
&= \sum_{X \subseteq E} \bar{\tau}(G/X, x) \bar{\varphi}(G|X, y).
\end{aligned}$$

The last equality follows from the decomposition (4.22) to the graph  $G/X$  and the decomposition (4.23) to the graph  $G|X$ .  $\square$

There are specific combinatorial interpretations on  $\kappa$  and  $\bar{\kappa}$  at some special integers, similar to that of Corollary 3.4. The special values of the Tutte polynomial

$T_G$  in the following corollary are observed directly by Gioan [16] by cycle-cocycle reversing systems.

**Corollary 4.8.** *Let  $[\mathcal{O}(G)]$ ,  $[\mathcal{O}_{AC}(G)]$ ,  $[\mathcal{O}_{TC}(G)]$ ,  $[\mathcal{O}_{CU}(G)]$ , and  $[\mathcal{O}_{EU}(G)]$  denote the sets of cut-Eulerian equivalence classes of  $\mathcal{O}(G)$ ,  $\mathcal{O}_{AC}(G)$ ,  $\mathcal{O}_{TC}(G)$ ,  $\mathcal{O}_{CU}(G)$ , and  $\mathcal{O}_{EU}(G)$  respectively. Then*

$$\begin{aligned} T_G(0, 0) &= \bar{\kappa}(G; -1, -1) = \kappa(G; 1, 1) = 0, \\ T_G(1, 1) &= \bar{\kappa}(G; 0, 0) = \#[\mathcal{O}(G)], \\ T_G(2, 2) &= \bar{\kappa}(G; 1, 1) = \#\mathcal{O}(G), \\ \kappa(G; 2, 2) &= \#[\mathcal{O}_{CE}(G)]; \end{aligned}$$

$$\begin{aligned} |T_G(0, -1)| &= |\bar{\kappa}(G; -1, -2)| = \kappa(G; 1, 2) = \#[\mathcal{O}_{EU}(G)], \\ |T_G(-1, 0)| &= |\bar{\kappa}(G; -2, -1)| = \kappa(G; 2, 1) = \#[\mathcal{O}_{CU}(G)]; \end{aligned}$$

$$\begin{aligned} T_G(1, 0) &= \bar{\kappa}(G; 0, -1) = |\kappa(G; 0, 1)| = \#[\mathcal{O}_{AC}(G)], \\ T_G(0, 1) &= \bar{\kappa}(G; -1, 0) = |\kappa(G; 1, 0)| = \#[\mathcal{O}_{TC}(G)]. \end{aligned}$$

Let  $[\mathcal{O}(G)]_{CU}$  and  $[\mathcal{O}(G)]_{EU}$  denote the sets of equivalence classes of  $\mathcal{O}(G)$  under the cut equivalence and Eulerian equivalence relations respectively. Then

$$\begin{aligned} T_G(1, 2) &= \bar{\kappa}(G; 0, 1) = \#[\mathcal{O}(G)]_{CU}, \\ T_G(2, 1) &= \bar{\kappa}(G; 1, 0) = \#[\mathcal{O}(G)]_{EU}. \end{aligned}$$

PROOF. It is completely parallel to the proof of Corollary 3.4 by modifying the concerned orientations to proper equivalence classes of those orientations, except the following two equalities:

$$\begin{aligned} \#[\mathcal{O}(G)]_{CU} &= \sum_{\rho \in \text{Rep}[\mathcal{O}(G)]} \#[\rho]_{EU}, \\ \#[\mathcal{O}(G)]_{EU} &= \sum_{\rho \in \text{Rep}[\mathcal{O}(G)]} \#[\rho]_{CU}, \end{aligned}$$

where  $\text{Rep}[\mathcal{O}(G)]$  is a set of distinct representatives of cut-Eulerian equivalence classes in  $[\mathcal{O}(G)]$ . The two equalities follow respectively from the set equations:

$$\begin{aligned} [\mathcal{O}(G)]_{CU} &= \bigsqcup_{\rho \in \text{Rep}[\mathcal{O}(G)]} \{[\sigma]_{CU} : \sigma \in [\rho]_{EU}\}, \\ [\mathcal{O}(G)]_{EU} &= \bigsqcup_{\rho \in \text{Rep}[\mathcal{O}(G)]} \{[\sigma]_{EU} : \sigma \in [\rho]_{CU}\}. \end{aligned}$$

□

**Theorem 4.9.** *Integral-Modular Relations: For each orientation  $\rho \in \mathcal{O}(G)$ , let  $[\rho]$  denote the cut-Eulerian equivalence class of  $\rho$  in  $\mathcal{O}(G)$ . Then*

$$\kappa_{\mathbb{Z}}(G; x, y) = \sum_{[\rho] \in [\mathcal{O}(G)]} \#[\rho] \kappa_{\rho}(G; x, y), \quad (4.24)$$

$$\bar{\kappa}_{\mathbb{Z}}(G; x, y) = \sum_{[\rho] \in [\mathcal{O}(G)]} \#[\rho] \bar{\kappa}_{\rho}(G; x, y). \quad (4.25)$$

Furthermore, if  $\#[\rho]$  is constant for all  $\rho \in \mathcal{O}(G)$ , then

$$\kappa_{\mathbb{Z}}(G; x, y) = \#[\rho] \kappa(G; x, y), \quad \bar{\kappa}_{\mathbb{Z}}(G; x, y) = \#[\rho] \bar{\kappa}(G; x, y).$$

Let  $y = 1$  in Theorem 4.9. We obtain a relation between the integral tension polynomial  $\tau_{\mathbb{Z}}(G, x)$  and the modular tension polynomial  $\tau(G, x)$ .

**Corollary 4.10.** *For each orientation  $\rho \in \mathcal{O}_{\text{AC}}(G)$ , let  $[\rho]$  denote the set of cut-equivalence class of  $\rho$  in  $\mathcal{O}_{\text{AC}}(G)$ . Then*

$$\tau_{\mathbb{Z}}(G, x) = \sum_{[\rho] \in [\mathcal{O}_{\text{AC}}(G)]} \#[\rho] \tau_{\rho}(G, x).$$

Furthermore, if  $\#[\rho]$  is constant for all  $\rho \in \mathcal{O}_{\text{AC}}(G)$ , then  $\tau_{\mathbb{Z}}(G, x) = \#[\rho] \tau(G, x)$ .

Let  $x = 1$  in Theorem 4.9. We obtain a relation between the integral flow polynomial  $\varphi_{\mathbb{Z}}(G, y)$  and the modular flow polynomial  $\varphi(G, y)$ , which answers the question asked by Beck and Zaslavsky [3].

**Corollary 4.11.** *For each orientation  $\rho \in \mathcal{O}_{\text{TC}}(G)$ , let  $[\rho]$  denote the set of Eulerian-equivalence class of  $\rho$  in  $\mathcal{O}_{\text{TC}}(G)$ . Then*

$$\varphi_{\mathbb{Z}}(G, y) = \sum_{[\rho] \in [\mathcal{O}_{\text{TC}}(G)]} \#[\rho] \varphi_{\rho}(G, y).$$

Furthermore, if  $\#[\rho]$  is constant for all  $\rho \in \mathcal{O}_{\text{TC}}(G)$ , then  $\varphi_{\mathbb{Z}}(G, y) = \#[\rho] \varphi(G, y)$ .

## 5. Connection to the Tutte polynomial

Recall that the *rank generating polynomial* (see [5], p.337) of a graph  $G$  is defined as

$$R_G(x, y) := \sum_{X \subseteq E} x^{r(E) - r(X)} y^{n(X)}, \quad (5.1)$$

The *Tutte polynomial* of  $G$  (see [5], p.339) is defined by shifting each variable one unit as

$$T_G(x, y) := R_G(x - 1, y - 1), \quad (5.2)$$

and satisfies the recurrence relation:

$$T_{\emptyset_n}(x, y) = 1, \\ T_G(x, y) = \begin{cases} x T_{G/e}(x, y) & \text{if } e \text{ is a bridge,} \\ y T_{G-e}(x, y) & \text{if } e \text{ is a loop,} \\ T_{G-e}(x, y) + T_{G/e}(x, y) & \text{otherwise,} \end{cases}$$

where  $\emptyset_n$  are the graphs with  $n$  vertices and no edges,  $G - e$  is the graph obtained from  $G$  by deleting the edge  $e$ , and  $G/e$  the graph obtained from  $G$  by contracting the edge  $e$ .

The following proposition is straightforward. The formula (5.3) is partly due to Reiner for  $p, q$  to be prime powers (see [25], Theorem 1); however, his statement is more general on a matroid of an integral matrix. The formula (5.4) is fully due to Reiner (see [25], Corollary 2).

**Proposition 5.1.** *Let  $A, B$  be finite abelian groups of orders  $|A| = p$ ,  $|B| = q$ , and  $\Omega := \Omega(G, \varepsilon; A, B)$ . Then  $R_G(x, y)$  has the following combinatorial interpretations:*

$$R_G(p, q) = \sum_{\substack{(f, g) \in \Omega \\ \text{supp } g \subseteq \ker f}} 2^{|\ker f - \text{supp } g|}, \quad (5.3)$$

$$R_G(-p, -q) = (-1)^{r(G)} \sum_{\substack{(f, g) \in \Omega \\ \text{supp } g = \ker f}} (-1)^{|\text{supp } g|}. \quad (5.4)$$

PROOF. For subsets  $X, Y \subseteq E$ , let  $\Omega_{X,Y} = \Omega_{X,Y}(G, \varepsilon; A, B)$  denote the set of pairs  $(f, g) \in \Omega$  such that  $f|_X = 0, g|_Y = 0$ . Then  $|\Omega_{X,X^c}| = p^{r\langle E \rangle - r\langle X \rangle} q^{n\langle X \rangle}$ . Thus

$$\begin{aligned}
 R_G(p, q) &= \sum_{X \subseteq E} |\Omega_{X,X^c}(G, \varepsilon; A, B)| \\
 &= \sum_{X \subseteq E} \sum_{\substack{(f,g) \in \Omega \\ \text{supp } g \subseteq X \subseteq \ker f}} 1 \\
 &= \sum_{\substack{(f,g) \in \Omega \\ \text{supp } g \subseteq \ker f}} \sum_{\text{supp } g \subseteq X \subseteq \ker f} 1 \\
 &= \sum_{\substack{(f,g) \in \Omega \\ \text{supp } g \subseteq \ker f}} 2^{|\ker f - \text{supp } g|}.
 \end{aligned}$$

Replace  $x$  by  $-p$  and  $y$  by  $-q$  in (5.1) and note that  $|X| = r\langle X \rangle + n\langle X \rangle$  for  $X \subseteq E$ . We obtain

$$\begin{aligned}
 R_G(-p, -q) &= (-1)^{r(G)} \sum_{X \subseteq E} (-1)^{|X|} p^{r\langle E \rangle - r\langle X \rangle} q^{n\langle X \rangle} \\
 &= (-1)^{r(G)} \sum_{X \subseteq E} (-1)^{|X|} \sum_{\substack{(f,g) \in \Omega \\ \text{supp } g \subseteq X \subseteq \ker f}} 1 \\
 &= (-1)^{r(G)} \sum_{\substack{(f,g) \in \Omega \\ \text{supp } g \subseteq \ker f}} \sum_{\text{supp } g \subseteq X \subseteq \ker f} (-1)^{|X|}.
 \end{aligned}$$

Using the Binomial Theorem, it is easy to see that

$$\sum_{\text{supp } g \subseteq X \subseteq \ker f} (-1)^{|X|} = (-1)^{|\ker f|} \delta_{\text{supp } g, \ker f},$$

where  $\delta_{X,Y} = 1$  if  $X = Y$  and  $\delta_{X,Y} = 0$  otherwise. Therefore we have

$$\begin{aligned}
 R_G(-p, -q) &= (-1)^{r(G)} \sum_{\substack{(f,g) \in \Omega \\ \text{supp } g \subseteq \ker f}} (-1)^{|\ker f|} \delta_{\text{supp } g, \ker f} \\
 &= (-1)^{r(G)} \sum_{\substack{(f,g) \in \Omega \\ \text{supp } g = \ker f}} (-1)^{|\text{supp } g|}.
 \end{aligned}$$

□

**Theorem 5.2.** *The rank generating polynomial  $R_G$  has the following combinatorial interpretation:*

$$R_G(x, y) = \sum_{[\rho] \in [\mathcal{O}(G)]} \bar{\kappa}_\rho(G; x, y), \quad (5.5)$$

$$R_G(-x, -y) = (-1)^{r(G)} \sum_{[\rho] \in [\mathcal{O}(G)]} (-1)^{|E(C_\rho)|} \kappa_\rho(G; x, y). \quad (5.6)$$

PROOF. Note that a real-valued tension-flow  $(f, g)$  of  $(G, \varepsilon)$  is complementary if and only if  $\text{supp } g = \ker f$ . Thus

$$K(G, \varepsilon; \mathbb{Z}_p, \mathbb{Z}_q) = \{(f, g) \in \Omega(G, \varepsilon; \mathbb{Z}_p, \mathbb{Z}_q) \mid \text{supp } g = \ker f\}$$



and (5.4) becomes

$$R_G(-p, -q) = (-1)^{r(G)} \sum_{(f,g) \in K(G, \varepsilon; \mathbb{Z}_p, \mathbb{Z}_q)} (-1)^{|\text{supp } g|}.$$

Applying the disjoint decomposition (4.12), we obtain

$$R_G(-p, -q) = \sum_{[\rho] \in [\mathcal{O}(G)]} (-1)^{r(G)} \sum_{(f,g) \in K_\rho(G, \varepsilon; \mathbb{Z}_p, \mathbb{Z}_q)} (-1)^{|\text{supp } g|}.$$

where  $K_\rho(G, \varepsilon; \mathbb{Z}_p, \mathbb{Z}_q) := \text{Mod}_{p,q}(\mathbb{Z}^2)^E \cap (p, q)\Delta_{\text{CTF}}^\rho(G, \varepsilon)$ . For each element  $(f, g)$  of  $K_\rho(G, \varepsilon; \mathbb{Z}_p, \mathbb{Z}_q)$ , let  $(\tilde{f}, \tilde{g})$  be an element of  $(\mathbb{Z}^2)^E \cap (p, q)\Delta_{\text{CTF}}^\rho(G, \varepsilon)$  such that  $\text{Mod}_{p,q}(\tilde{f}, \tilde{g}) = (f, g)$  by Lemma 4.4. Then  $P_{\rho, \varepsilon}(\tilde{f}, \tilde{g}) \in \Delta_{\text{CTF}}^+(G, \rho)$ . Since  $P_{\rho, \varepsilon}(\tilde{f} + \tilde{g}) = [\rho, \varepsilon](\tilde{f} + \tilde{g}) > 0$ , it is clear that  $\text{supp } [\rho, \varepsilon]\tilde{g} = E(C_\rho)$ . Since  $\text{supp } [\rho, \varepsilon]\tilde{g} = \text{supp } \tilde{g} = \text{supp } g$ , then  $\text{supp } g = E(C_\rho)$ . Thus

$$R_G(-p, -q) = (-1)^{r(G)} \sum_{[\rho] \in [\mathcal{O}(G)]} (-1)^{|E(C_\rho)|} \#K_\rho(G, \varepsilon; \mathbb{Z}_p, \mathbb{Z}_q).$$

Apply (4.10); we obtain (5.6) immediately. The formula (5.5) follows from the Reciprocity Law (3.9) on  $\kappa_\rho$  and  $\bar{\kappa}_\rho$ .  $\square$

#### PROOF OF THEOREM 1.3 AND COROLLARY 1.4

Theorem 1.3 follows from (5.5) and (1.20). Corollary 1.4 follows from the definition (5.2) of  $T_G$  and the fact that a nonnegative, integer-valued,  $(p, q)$ -tension-flow  $(f, g)$  is an integer-valued tension-flow such that  $0 \leq f(e) \leq p-1$ ,  $0 \leq g(e) \leq q-1$  for  $e \in E$ .  $\square$

**Example 5.3.** Let  $G$  be the graph in the following Figure 1. Its Tutte polynomial is given by deletion-contraction as

$$T(x, y) = y^3 + x^2 + 2xy + 2y^2 + x + y.$$

Let  $(x_i)$  and  $(y_i)$  ( $1 \leq i \leq 5$ ) be tensions and flows of  $G$  with the given orientation in Figure 1. The complementary tension-flows  $(x_i, y_i)$  satisfy the following system of linear equations and inequalities:

$$\begin{aligned} x_1 - x_2 - x_3 &= 0, & y_2 - y_3 + y_4 - y_5 &= 0, & x_i y_i &= 0, \\ x_2 - x_4 &= 0, & y_1 + y_2 + y_4 &= 0, & x_i + y_i &\neq 0, \\ x_3 - x_5 &= 0, & y_1 + y_3 + y_5 &= 0, & 1 \leq i &\leq 5. \end{aligned}$$

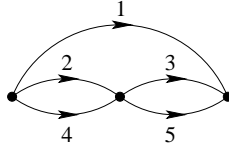


FIGURE 1. The edge-labeled graph  $G$ .

When  $(x_i)$  is over an abelian group  $A$  and  $(y_i)$  over another abelian group  $B$ , the conditions  $x_i y_i = 0$  and  $x_i + y_i \neq 0$  are understood as  $\text{supp } (x_i) = \ker(y_i)$ . Let  $|A| = p$  and  $|B| = q$ ; or set  $|x_i| < p$  and  $|y_i| < q$  over  $\mathbb{Z}$ . By counting the number

of solutions of the above system, we obtain the complementary polynomial  $\kappa$  and integral complementary polynomial  $\kappa_{\mathbb{Z}}$  of  $G$  as follows:

$$\kappa(p, q) = (p-1)(p-2) + 2(p-1)(q-1) + (q-1)(q-2)^2;$$

$$\begin{aligned} \kappa_{\mathbb{Z}}(p, q) = & 3(p-1)(p-2) + 8(p-1)(q-1) \\ & + 2(q-1)(q-3)(2q-3) + \frac{(q-1)q(2q-1)}{3}. \end{aligned}$$

There are 8 ( $= T(1, 1)$ ) cut-Eulerian equivalence classes of orientations, 2 ( $= T(1, 0)$ ) cut equivalence classes of cyclic orientations, and 4 ( $= T(0, 1)$ ) Eulerian equivalence classes of totally cyclic orientations. There are 24 ( $= T(1, 2)$ ) cut equivalence classes of orientations (2, 4, and 18 in Figures 2, 3, and 4 respectively). There are 14 ( $= T(2, 1)$ ) Eulerian equivalence classes of orientations (6, 4, and 4 in Figures 2, 3, and 4 respectively). There are 32 ( $= T(2, 2)$ ) total number of orientations. However, there are essentially only 4 different local complementary polynomials:

$$\kappa_1(p, q) = \frac{(p-1)(p-2)}{2},$$

$$\kappa_2(p, q) = (p-1)(q-1),$$

$$\kappa_3(p, q) = -(q-1)^2 + \frac{q^2(q-1)}{2} - \frac{(q-1)q(2q-1)}{6},$$

$$\kappa_4(p, q) = -(q-1)^2 + \frac{(q-1)q(2q-1)}{6}.$$

The dual complementary (also rank generating) polynomial  $\bar{\kappa}$  and the dual integral complementary polynomial  $\bar{\kappa}_{\mathbb{Z}}$  are subsequently obtained as

$$\begin{aligned} \bar{\kappa}(p, q) &= 2[\kappa_1(-p, -q) + \kappa_2(-p, -q) - \kappa_3(-p, -q) - \kappa_4(-p, -q)] \\ &= (p+1)(p+2) + 2(p+1)(q+1) + 4(q+1)^2 + q^2(q+1) \\ &= q^3 + p^2 + 2pq + 5q^2 + 5p + 10q + 8 \\ &= T(p+1, q+1); \end{aligned}$$

$$\begin{aligned} \bar{\kappa}_{\mathbb{Z}}(p, q) &= 6\kappa_1(-p, -q) + 8\kappa_2(-p, -q) - 8\kappa_3(-p, -q) - 10\kappa_4(-p, -q) \\ &= 3(p+1)(p+2) + 8(p+1)(q+1) \\ &\quad + 18(q+1)^2 + 4q^2(q+1) + \frac{q(q+1)(2q+1)}{3}. \end{aligned}$$

Other special values of the polynomials  $\kappa$ ,  $\bar{\kappa}$ ,  $\kappa_{\mathbb{Z}}$ , and  $\bar{\kappa}_{\mathbb{Z}}$  are

$$\kappa(2, 1) = T(-1, 0) = \#[\mathcal{O}_{\text{CU}}] = 0,$$

$$\kappa(1, 2) = T(0, -1) = \#[\mathcal{O}_{\text{EU}}] = 0,$$

$$\kappa(2, 2) = \#[\mathcal{O}_{\text{CE}}] = 0.$$

It means that  $G$  is *not* a cut (equivalently not bipartite), *not* an Eulerian graph, and *not* a disjoint union of a cut and an Eulerian subgraph. The eight cut-equivalence classes of orientations are exhibited as follows:

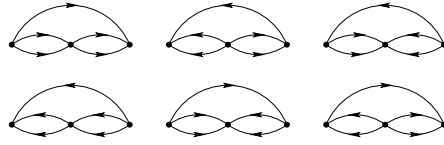


FIGURE 2. Two ( $= T(1, 0)$ ) cut equivalence classes of acyclic orientations have the same local complementary polynomial  $\kappa_1$ .

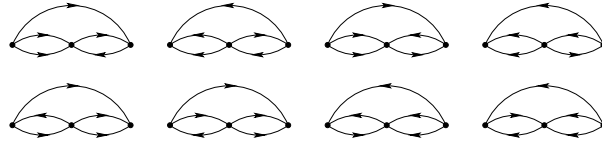


FIGURE 3. Two ( $= T(1, 1) - T(1, 0) - T(0, 1)$ ) cut-Eulerian equivalence classes of cyclic but not totally cyclic orientations have the same local complementary polynomial  $\kappa_2$ .

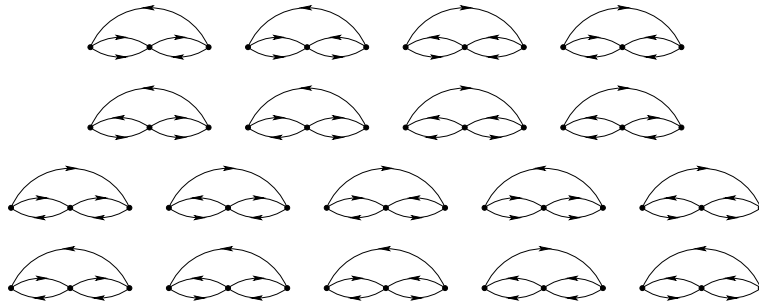


FIGURE 4. Four ( $= T(0, 1)$ ) Eulerian equivalence classes of totally cyclic orientations; the first two and the last two classes have the same local complementary polynomial  $\kappa_3$  and  $\kappa_4$ , respectively.

## References

- [1] A. Barg, The matroid of supports of a linear code, *Appl. Algebra Eng. Commun. Comput.* **8** (1977), 165–172.
- [2] A. Barvinok, *A Course in Convexity*, Graduate Studies in Mathematics, **54**, Amer. Math. Soc., Providence, RI 2002.
- [3] M. Beck and T. Zaslavsky, The number of nowhere-zero flows of graphs and signed graphs, *J. Combin. Theory Ser. B* **96** (2006), 901–918.
- [4] M. Beck and S. Robins, *Computing the Continuous discretely: Integer-Point Enumeration in Polyhedra*. Undergraduate Texts in Mathematics, Springer, 2007.
- [5] B. Bollobás, *Modern Graph Theory*, Springer, 2002.
- [6] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Elsevier Science Publishing Co., 1976.
- [7] F. Breuer and R. Sanyal, Ehrhart theory, modular folow reciprocity, and the Tutte polynomial, arXiv.0907.0845v1 [math.CO] 5 July 2009.
- [8] T. Brylawski and J.G. Oxley, The Tutte polynomial and its applications, in: *Matroid Applications*, N. White (ed.), Encyclopedia of Mathematics and Its Applications, Vol. **40**, Cambridge Univ. Press, 1992.
- [9] Chang, H.; Ma, J.; Yeh, Y.-N. Tutte polynomials and  $G$ -parking functions, *Adv. in Appl. Math.* **44** (2010), 231–242.

- [10] B. Chen, Lattice points, Dedekind sums, and Ehrhart polynomials of lattice polyhedra, *Discrete Comput. Geom.* **28** (2002), 175–199.
- [11] B. Chen, Ehrhart polynomials of lattice polyhedral functions, in: *Integer Points in Polyhedra – Geometry, Number Theory, Algebra, Optimization*, 37–63, Contemp. Math., **374**, Amer. Math. Soc., Providence, RI, 2005.
- [12] B. Chen, Orientations, lattice polytopes, and group arrangements I: Chromatic and tension polynomials of graphs, *Ann. Comb.* **13** (2010), 425–452.
- [13] B. Chen and R. Stanley, Orientations, lattice polytopes, and group arrangements II: Modular and integral flow polynomials of graphs, preprint.
- [14] B. Chen, Conference abstract, 2007 International Conference on Graph Theory and Combinatorics & Fourth Cross-strait Conference on Graph Theory and Combinatorics, June 24–29, 2007.
- [15] H. Crapo and G.-C. Rota, *On the Foundations of Combinatorial Theory: Combinatorial Geometries*, MIT Press, Cambridge, MA, 1970.
- [16] E. Gioan, Enumerating degree sequences in digraphs and a cycle-cocycle reversing system, *European J. Combin.* **28** (2007), 1351–1366.
- [17] C. Greene, Weight enumeration and the geometry of linear codes, *Stud. Appl. Math.* **55** (1976), 119–128.
- [18] C. Greene and T. Zaslavsky, On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs, *Trans. Amer. Math. Soc.* **280** (1983), 97–126.
- [19] F. Jaeger, On Tutte polynomials of matroids representable over  $GF(q)$ , *European J. Combin.* **10** (1989), 247–255.
- [20] M. Kochol, Tension polynomials of graphs, *J. Graph Theory* **40** (2002), 137–146.
- [21] M. Kochol, Polynomials associated with nowhere-zero flows, *J. Combin. Theory Ser. B* **84** (2002), 260–269.
- [22] M. Kochol, Tension-flow polynomials on graphs, *Discrete Math.* **274** (2004), 173–185.
- [23] W. Kook, V. Reiner, and D. Stanton, A convolution formula for the Tutte polynomial, *J. Combin. Theory Ser. B* **76** (1999), 297–300.
- [24] G.J. Minty, Monotone networks, *Proc. Roy. Soc. London. Ser. A* **257** (1960), 194–212.  
formula for the Tutte polynomial, *J. Combin. Theory Ser. B* **76** (1999), 297–300.
- [25] V. Reiner, An interpretation for the Tutte polynomial, *European J. Combin.* **20** (1999), 149–161.
- [26] R.P. Stanley, Acyclic orientations of graphs, *Discrete Math.* **5** (1973), 171–178.
- [27] R.P. Stanley, *Enumerative Combinatorics I*, Cambridge Univ. Press, 1997.
- [28] W.T. Tutte, A ring in graph theory, *Proc. Camb. Phil. Soc.* **43** (1947), 26–40.
- [29] C.-Q. Zhang, *Integer Flows and Cycle Covers of Graphs*, Marcel Dekker, New York, NY, 1997.
- [30] D.J.A. Welsh, *Complexity: Knots, Colourings and Counting*, Cambridge Univ. Press, Cambridge, 1993.
- [31] C.-Q. Zhang, *Integer Flows and Cycle Covers of Graphs*, Marcel Dekker, New York, NY, 1997.

DEPARTMENT OF MATHEMATICS, HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY,  
CLEAR WATER BAY, KOWLOON, HONG KONG  
E-mail address: mabfchen@ust.hk